# What do we know about Betti Numbers?

$R = k[x,y,z]$ $\qquad$ $I = (xy, yz, xz)$

$$\begin{tikzcd}
0 \arrow[r] & R^2 \arrow[rr, "\begin{bmatrix} z & 0 \\ -x & x \\ 0 & -z \end{bmatrix}"] \arrow[dr, two heads] & & R^3 \arrow[r, "{[xy\ yz\ xz]}"] & R^1 \arrow[r] & R/(xy, yz, xz) \\
 & & K \arrow[ur, hook] & & &
\end{tikzcd}$$

$z(xy) + (-x)(yz)$

Betti #s of $(R/I)$ : $\beta_0 = 1$, $\beta_1 = 3$, $\beta_2 = 2$

$R = k[x,y,u,v]$ $\qquad$ $I = (x^2, y^2, ux+vy)$

$$R^5 \xrightarrow{\begin{bmatrix} -y^2 & 0 & -ux-vy & -yu & -u^2 \\ x^2 & -ux-vy & 0 & -xv & v^2 \\ 0 & y^2 & x^2 & xy & xu-yv \end{bmatrix}} R^3 \xrightarrow{[x^2\ \ y^2\ \ ux+vy]} R \longrightarrow R/I$$

$$\begin{array}{ccccc} u & x & 0 & -y & 0 \end{array}$$

$$0 \to R^2 \to R^4 \longrightarrow R^5 \longrightarrow R^3 \longrightarrow R \to R/I$$

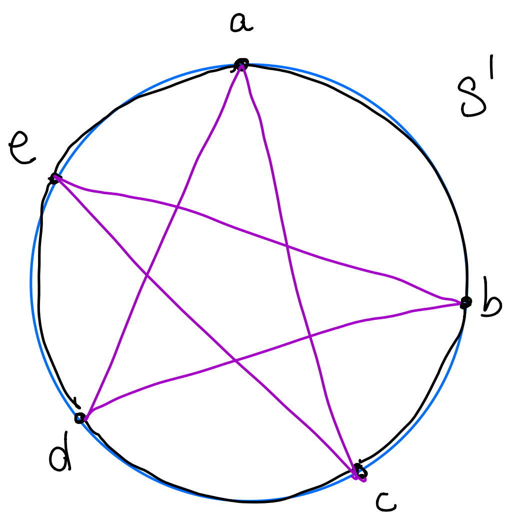
deg 0 $\quad$ deg 2

$\beta_0 = 1$, $\beta_1 = 3$, $\beta_2 = 5$, $\beta_3 = 4$, $\beta_4 = 2$

Thm (Burch–Bruns) $\forall N, M$ $\exists$ Ideal $I$ with 3 generators:

$I = (f_1, f_2, f_3)$ : $\qquad \beta_2, \qquad > N.$

Where $I$ is homogeneous $I \subset k[x_1, \dots, x_D]$ D large.




Non-edges

$$I = (ac, ad, bd, be, ce)$$

$$R^1 \to R^5 \to R^5 \to R \to R/I$$

$\beta_0 = 1$ start deg 0

$\beta_1 = 5$ deg 2

$\beta_2 = 5$ deg 3

$\beta_3 = 1$ deg 5

Macaulay 2:

```
i19 : R = QQ[a..e]; C = res ideal"ac,ad,bd,be,ce"; C.dd

             1                          5
o21 = 0 : R  <------------------------ R  : 1
             | ac ad bd be ce |

             5                            5
      1 : R  <---------------------------- R  : 2
             {2} | 0  -d -e 0  0  |
             {2} | -b c  0  0  0  |
             {2} | a  0  0  0  -e |
             {2} | 0  0  0  -c d  |
             {2} | 0  0  a  b  0  |

             5                  1
      2 : R  <---------------- R  : 3
             {3} | -ce |
             {3} | -be |
             {3} | bd  |
             {3} | -ad |
             {3} | -ac |

             1
      3 : R  <----- 0 : 4
             0
```

deg 2

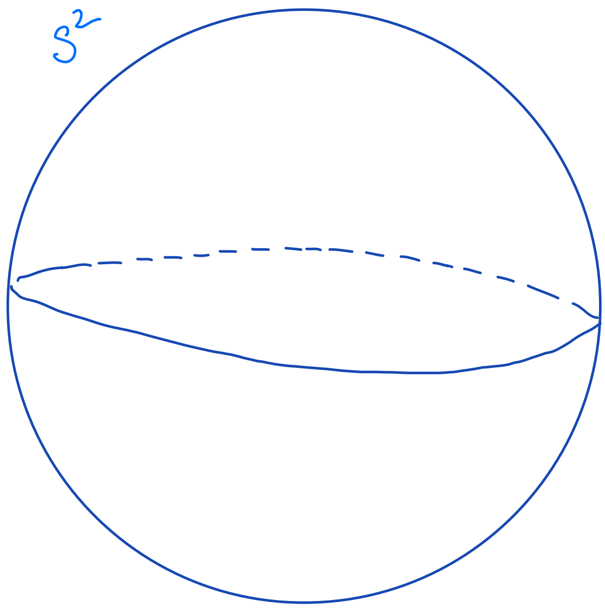
deg 3

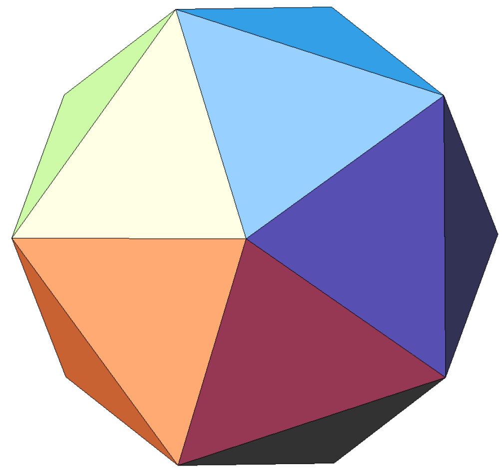
deg 5

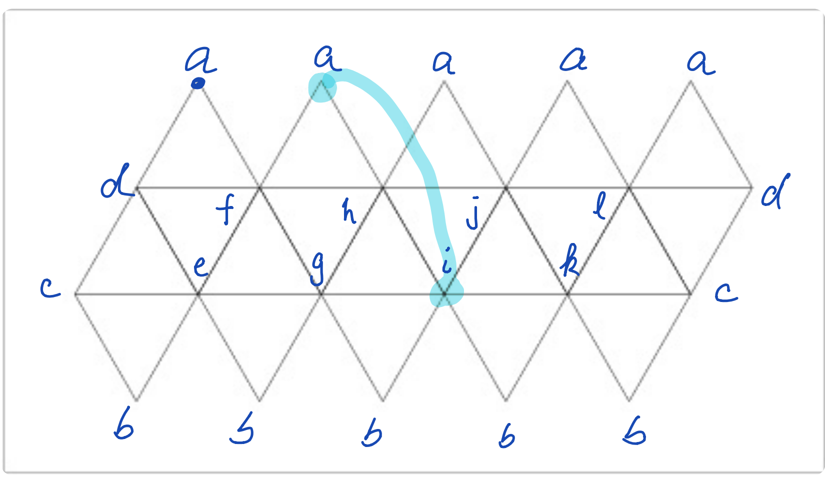
$$-ce\begin{bmatrix} 0 \\ -b \\ a \\ 0 \\ 0 \end{bmatrix} + \cdots$$

deg 3

$$\underbrace{-d(ac)+c(ad)}_{\text{deg } 3}=0$$






12 vertices
30 edges
20 faces

$\binom{12}{2} = 66$

Missing edges: "36 missing edges"

```
i25 : R = QQ[a..l]; C = res ideal"ab,ac,ae,ag,ai,ak
,bd,bf,bh,bj,bl,cf,cg,ch,ci,cj,dg,dh,di,dj,dk,eh,ei
,ej,ek,el,fi,fj,fk,fl,gj,gk,gl,hk,hl,il"; betti C

            0  1   2   3   4   5   6   7  8 9
o27 = total: 1 36 160 327 412 412 327 160 36 1
         0: 1  .   .   .   .   .   .   .  . .
         1: . 36 160 315 300 112  12   .  . .
         2: .  .   .  12 112 300 315 160 36 .
         3: .  .   .   .   .   .   .   .  . 1

o27 : BettiTally
```

$\beta_0 = 1$

$\beta_1 = 36$ deg 2

$\beta_2 = 160$ deg 3

$\beta_3 = 315$ deg 4
$12$ deg 5

$\beta_4 = 300$
$112$

$\vdots$

$\beta_9 = 1$ deg 12

polynomial : $\sum \underset{\text{numbers}}{\text{betti}} (-1)^i t^{\deg}$

$$= 1t^0 - 36t^2 + 160t^3 - 315t^4 - 12t^5 + \cdots$$

$$\cdots -1t^{12}$$

```
i38 : ZZ[t];  p = 1-36*t^2+160*t^3-315*t^4-12*t^5+3
00*t^5+112*t^6-112*t^6-300*t^7+12*t^7+315*t^8-160*t
^9+36*t^10-1*t^12; factor(p)

             9           2
o40 = (t - 1) (t + 1)(t  + 8t + 1)(-1)

o40 : Expression of class Product
```

$$(t+1)(t^2+8t+1)$$

$$= t^3 + 9t^2 + 9t + 1$$

Magic:

$$= (1+m)^3 + 9(1+m)^2 + 9(1+m) + 1$$

| 1 | $m$ | $m^2$ | $m^3$ |
|---|---|---|---|
| 1 | 3 | 3 | 1 |
| 9( 1 | 2 | 1 ) | |
| 9( 1 | 1 ) | | |
| 1 | | | |

$$20 + 30m + 12m^2 + 1m^3$$

f-vector for Icosahedron!

Triangulation of any Manifold $X$ using vertices, edges, faces $f_0, f_1, f_2, \ldots$

Ideal $I$ generated by monomials corresponding to non-faces

Betti Numbers of $R/I$

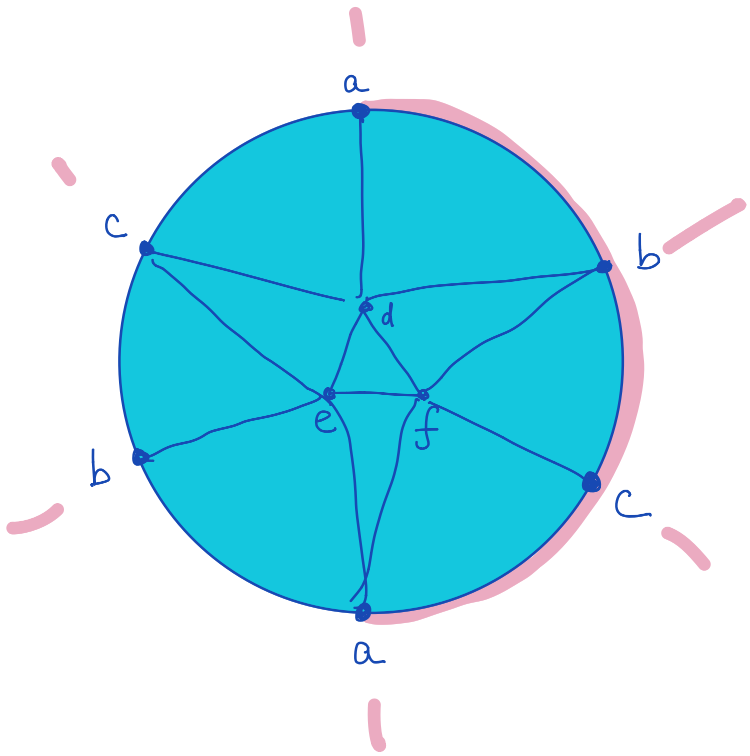
$$\frac{\sum \beta_{ij} (-1)^i t^j}{(1-t)^{\#\text{vertices}}}$$

$$\frac{h(t)}{(1-t)^{\dim X+1}}$$

Cool Theorem:

$S^1, S^2$

If $X$ is a sphere then the Betti numbers will be symmetric




Ideal: ( 10 non-faces )

```
i61 : R = QQ[a..f]; C = res ideal"abc,abf,ace,ade,adf,bcd,bde,bef,cdf,cef
"; C.dd

           1                                                 10
o63 = 0 : R  <---------------------------------------------- R   : 1
                | abc bcd ace ade bde abf adf cdf bef cef |

           10
15
      1 : R   <---------------------------------------------------------- R
   : 2
                  {3} | -d -e 0  0  0  -f 0  0  0  0  0  0  0  0  0  |
                  {3} | a  0  0  0  -e 0  0  0  -f 0  0  0  0  0  0  |
                  {3} | 0  b  0  -d 0  0  0  0  0  0  -f 0  0  0  0  |
                  {3} | 0  0  -b c  0  0  0  0  0  0  0  0  -f 0  0  |
                  {3} | 0  0  a  0  c  0  0  0  0  0  0  0  0  -f 0  |
                  {3} | 0  0  0  0  0  c  -d 0  0  -e 0  0  0  0  0  |
                  {3} | 0  0  0  0  0  0  b  -c 0  0  0  0  e  0  0  |
                  {3} | 0  0  0  0  0  0  0  a  b  0  0  0  0  0  -e |
                  {3} | 0  0  0  0  0  0  0  0  0  a  0  -c 0  d  0  |
                  {3} | 0  0  0  0  0  0  0  0  0  0  a  b  0  0  d  |

           15                           6
      2 : R   <------------------------ R   : 3
                  {4} | e  f  0  0  0  0  |
                  {4} | -d 0  -f 0  0  0  |
                  {4} | -c 0  0  f  0  0  |
                  {4} | -b 0  0  0  f  0  |
                  {4} | a  0  0  0  0  -f |
                  {4} | 0  -d e  0  0  0  |
                  {4} | 0  -c 0  e  0  0  |
                  {4} | 0  -b 0  0  e  0  |
                  {4} | 0  a  0  0  0  e  |
                  {4} | 0  0  c  -d 0  0  |
                  {4} | 0  0  -b 0  -d 0  |
                  {4} | 0  0  a  0  0  -d |
                  {4} | 0  0  0  -b c  0  |
                  {4} | 0  0  0  a  0  -c |
                  {4} | 0  0  0  0  a  b  |

           6
      3 : R  <----- 0 : 4
```

CM

```
i65 : R = ZZ/2[a..f]; C = res ideal"abc,abf,ace,ade,adf,bcd,bde,bef,cdf,c
ef"; C.dd

           1                                                10
o67 = 0 : R  <--------------------------------------------- R   : 1
               | abc bcd ace ade bde abf adf cdf bef cef |

           10                                          15
      1 : R   <--------------------------------------- R   : 2
               {3} | d e 0 0 0 f 0 0 0 0 0 0 0 0 0 |
               {3} | a 0 0 0 e 0 0 0 f 0 0 0 0 0 0 |
               {3} | 0 b 0 d 0 0 0 0 0 0 f 0 0 0 0 |
               {3} | 0 0 b c 0 0 0 0 0 0 0 0 f 0 0 |
               {3} | 0 0 a 0 c 0 0 0 0 0 0 0 0 f 0 |
               {3} | 0 0 0 0 0 c d 0 0 e 0 0 0 0 0 |
               {3} | 0 0 0 0 0 0 b c 0 0 0 0 e 0 0 |
               {3} | 0 0 0 0 0 0 0 a b 0 0 0 0 0 e |
               {3} | 0 0 0 0 0 0 0 0 0 a 0 c 0 d 0 |
               {3} | 0 0 0 0 0 0 0 0 0 0 a b 0 0 d |

           15                          7
      2 : R   <----------------------- R   : 3
               {4} | e f 0 0 0 0 0  |
               {4} | d 0 f 0 0 0 df |
               {4} | c 0 0 f 0 0 0  |
               {4} | b 0 0 0 f 0 bf |
               {4} | a 0 0 0 0 f 0  |
               {4} | 0 d e 0 0 0 de |
               {4} | 0 c 0 e 0 0 ce |
               {4} | 0 b 0 0 e 0 0  |
               {4} | 0 a 0 0 0 e 0  |
               {4} | 0 0 c d 0 0 0  |
               {4} | 0 0 b 0 d 0 0  |
               {4} | 0 0 a 0 0 d 0  |
               {4} | 0 0 0 b c 0 bc |
               {4} | 0 0 0 a 0 c 0  |
               {4} | 0 0 0 0 a b 0  |

           7              1
      3 : R  <----------- R  : 4
              {5} | f |
              {5} | e |
              {5} | d |
              {5} | c |
              {5} | b |
              {5} | a |
              {6} | 0 |

           1
      4 : R  <----- 0 : 5
              0
```

not CM

hf $\dfrac{1-10t^2+15t^3-6t^4}{(1-t)^6}$

dim, multiplicit.

$\dfrac{1-10t^2+15t^3-6t^4-t^5+t^5}{(1-t)^6}$

dim, mult,

Geometry of Syzygies - Eisenbud