How Big Are the Betti Numbers of Finite Length Modules?

October 23, 2020

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- I a homogeneous ideal.

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Recall,

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- $\beta_1(S/I) = \#$ generators of I
- $\beta_2(S/I) = \#$ relations on gens of I
- $\cdots$

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- $\beta_1(S/I) = \#$ generators of I
- $\cdots$
- $\beta_{pdim}(S/I) =$ last nonzero betti number.

Krull Altitude Theorem

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We will refer to $\mathrm{codim}(I)$ instead of height.

$$\beta_1(S/I) \geq \mathrm{codim}(I)$$

Auslander-Buchsbaum

$$\mathrm{pdim}(S/I) + \mathsf{depth}(S/I) = \mathsf{dim}\, k[x_1, \ldots, x_n]$$

$$\ldots$$

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$$\beta_{\mathrm{codim}(I)}(S/I) \quad \neq 0.$$

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$$\beta_{\mathsf{codim}(I)}(S/I) \quad \geq 1.$$

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- $\text{codim}(I) = c$.

What can we say about the Betti numbers of S/I?

Recall,

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- $\beta_1(S/I) \geq c$
- $\cdots$
- $\beta_c(S/I) \geq 1$.

General Case

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- $\beta_i(S/I) =?$
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If I is an ideal of codimension c then for all i:

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Known Results

- (Buchsbaum-Eisenbud '77) True if the resolution of S/I has a DG-Algebra structure
- True in general for $c \leq 4$ (these proofs are easy)
- Open for $c = 5$
  - Even open for ideals with 6 generators!
- True in other special cases
  - If I is monomial, licci, of "low regularity" etc.

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- Proven for $c = 5$ (1993 Avramov-Buchweitz)
- True for all c (if $\mathrm{char} k \neq 2$.) (Walker 2018)
- True in other special cases
  - If I is monomial, licci, of "low regularity" etc.
  - But these bounds are stronger!

Conjecture/Theorem (Walker 2018)

If I is an ideal of height c (and $\mathrm{char} k \neq 2$) then

$$\sum \beta_i(S/I) \geq 2^c$$

and

In char 2 the best lower bound is (Walker 2018)

$$2(\sqrt{3})^{c-1} > 2^{0.79c+0.208}$$

The Avramov-Buchweitz bound (1993)

$$(\sqrt{3})^c > 2^{0.79c}$$

only holds if the multiplicity of I is even but not divisible by 6.

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- If equality doesn't hold, how much bigger is $\sum \beta_i(S/I)$?
- Well, $\sum \beta_i(S/I)$ must be even.
- So if I is not generated by a regular sequence then $\sum \beta_i \geq 2^c + 2$.

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- Is there an ideal of codimension $c \geq 3$ with $\sum \beta_i(S/I) = 2^c + 2$? (I've not found one)

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(there are plenty of these)

Previous Results

$I \subset k[x_1, \ldots, x_n]$ not generated by a regular sequence

$$\text{is } \sum \beta_i(S/I) \geq 2^c + 2^{c-1} = 1.5(2^c)?$$

This is true when:

- For any ideal if $c \leq 4$
  - (Charalambous-Evans-Miller) Requires Classification of Tor Algebra Structures

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- (B-Seiner) I is monomial of height c, not gen. by reg. seq. then
  bound for β_i doesn't hold, yet still $\sum \beta_i(S/I) \geq 2^c + 2^{c-1}$.

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  (This is actually true if M or S/I is multigraded!)
- Or maybe $\beta_i \geq 1.5\binom{c}{i}$
- Or maybe if $\beta_i + \beta_{c-i} \geq 3\binom{c}{i}$
  (For instance, maybe the first half of the betti numbers are at least $2\binom{c}{i}$...)

Main Result

Main Theorem (B-Wigglesworth)

Let I be an ideal of height c and let a be the degree of the smallest generator of I. If $\mathrm{reg}(S/I) \leq 2a - 2$ then for all $c \geq 3$

$$\sum \beta_i(S/I) \geq 2^c + 2^{c-1}$$

and this is true because for all $c \geq 9$ the first half of the betti numbers satisfy: $\beta_i \geq 2\binom{c}{i}$ and the last half satisfy $\beta_i \geq \binom{c}{i}$.

A statement is also true for modules:

$$\sum \beta_i(M) \geq \beta_0(M)(2^c + 2^{c-1})$$

- Assume I has height c in $k[x_1,\dots,x_n]$
  - → And S/I is not a CI

	$c \leq 4$	$c \geq 5$	Monomial: Artinian $(c=n)$	Monomial: Non-Artinian $(c<n)$	low regularity
$\beta_i \geq \binom{c}{i}+\binom{c-1}{i-1}$	False	False	('91 CE)	False	False
$\sum \beta_i \geq (1.5)2^c$	('90 CEM) Classification of Tor Algebras	?	⇓	('18 B-Seiner) Delicate Splitting Argument	yes $\forall c$. ⇑ + check $\sum \beta_i$ for $5 \leq c \leq 8$
First half of $\beta_i \geq 2\binom{c}{i}$	False	False	False	False	True for $c \geq 9$