Combinatorial Commutative Algebra - I

10/30/20

Monomial ideals and their combinatorial structure

Outline:

- Taylor's resolution ⟶ non-minimal, every monomial ideal
  gcd properties easier to handle if I is squarefree
- Stanley-Reisner's theory for squarefree monomial ideals
  - simplicial complexes / graphs
    (vertices = variables, n-faces = squarefree generators of degree n)
  - Hochster / Eagon-Reiner formulas for graded Betti #'s
    (in terms of reduced simplicial cohomology)

References:

[Herzog-Hibi]: "Monomial ideals", GTM 260, Springer

[Francisco-Mermin-Schweigh]: "A survey on Stanley-Reisner Theory"
available on Chris Francisco's webpage

[Stanley]: Combinatorics and Commutative algebra

<u>Setting:</u> $S = k[X_1, \dots, X_n]$, I S-ideal

- homogeneous if $I = (f_1, f_2, \dots, f_m)$ and the f_i are <u>homogeneous</u> polynomials of degrees d_i

  E.g. $f(x_1, x_2, x_3) = \underbrace{3x_1^2 x_2}_{\text{homog. of degree } 3} + \underbrace{4x_3}_{\text{homog. of degree } 1}$ NOT homogeneous

- monomial if $I = (u_1, \dots, u_m)$ and the u_i are <u>monomials</u> of degrees d_i $\quad u_i = \underbrace{r_i}_{\text{coefficient.}} x_1^{a_1} \cdots x_n^{a_n} \qquad a_1 + \dots + a_n = d_i \qquad a_i \geq 0$

  Notice: monomial ideals are homogeneous

- squarefree monomial if $I = (u_1, \dots, u_m)$ and the u_i are squarefree monomials, that is: for all i if $x_j \mid u_i$ then $x_j^2 \nmid u_i$

  $I = (x_1 x_2, x_2 x_3, x_4^2)$ not squarefree

  $J = (x_1 x_2, x_2 x_3, x_4 x_5)$ squarefree.

<u>Our goals:</u>

- Understand free resolutions & Betti numbers of monomial ideals
- Figure out "how special" monomial ideals are among all homogeneous ideals, and "how special" squarefree monomial ideals are among all monomial ideals.

starting next week

Taylor resolution

$S = k[x_1, \dots, x_n]$, $\quad I = \underbrace{(u_1, \dots, u_m)}_{\text{irredundant generators}}$ monomial ideal in S

Find relations among the generators

$$S^m \xrightarrow{\partial_1} I$$
$$e_i \longmapsto u_i$$

e_i: free basis of S^m

For each u_i, u_j define $u_{i,j} := \dfrac{u_j}{\gcd(u_i, u_j)}$

Notice: $\quad u_i \, u_{ij} = u_j \, u_{ji} = \dfrac{u_i u_j}{\gcd(u_i, u_j)}$

This means that $\quad u_{ij} \, u_i - u_{ji} \, u_j = 0$

So, $u_{ij} e_i - u_{ji} u_j$ gives the syzygy between u_i and u_j

$$u_{i_1, \dots, i_k} = \frac{u_{i_1, \dots, i_{k-2}, i_k}}{\gcd(u_{i_1, \dots, i_{k-1}}, u_{i_1, \dots, i_{k-2}, i_k})}$$

[Taylor, 1968]: $I = (u_1, \dots, u_m)$ monomial ideal. A free resolution for I is

$$0 \to F_m \to \cdots \to F_j \xrightarrow{\partial_j} \cdots \to F_1 \xrightarrow{\partial_1} I \to 0 \quad \text{where}$$

- $F_j = R^{\binom{m}{j}}$ $\to$ same free modules as in the Koszul complex
- $K_j = \ker(\partial_j)$ is the submodule of F_j generated by the $\binom{m}{j+1}$ elements
$$b_{i_1 \cdots i_{j+1}} = \sum_{k=1}^{j+1} (-1)^{k+1} u_{i_1, \dots, \hat{i_k}, \dots, i_{j+1}, i_k} \underbrace{e_{i_1 \cdots \hat{i_k} \cdots i_{j+1}}}_{\text{basis element of } F_j}$$
- $\partial_j(e_{i_1, \dots, i_{j+1}}) = \displaystyle\sum_{k=1}^{j+1} (-1)^{k+1} u_{i_k} \, e_{i_1 \cdots \hat{i_k} \cdots i_{j+1}}$

Caution: Koszul complex (when exact) gives a minimal free resolution

Taylor's Resolution is not minimal.

Combinatorial structure

$I = (u_1, \dots, u_m)$ squarefree monomial ideal in $k[x_1, \dots, x_n]$

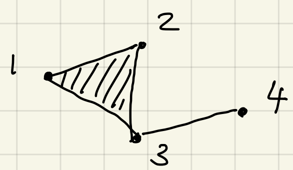
↳ minimal generating set for I

$n = \#$ variables $\longleftrightarrow$ $[n] = \{1, \dots, n\}$

$u_i = x_1 x_2 x_3$ $\longleftrightarrow$ "triangle" of vertices 1, 2, 3

$u_j = x_3 x_4$ $\longleftrightarrow$ "edge" joining 3 and 4

$I = (x_3 x_4, x_1 x_2 x_3)$ $\longleftrightarrow$



In fact, there is a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{squarefree monomial} \\ \text{ideals in } k[x_1, \dots, x_n] \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{simplicial complexes} \\ \text{on } [n] \end{array} \right\}$$

Def: $\Delta \subseteq \mathcal{P}([n])$ is a simplicial complex if for each $F \in \Delta$ every $F' \subseteq F$ is in Δ as well.

faces

0-dim face $\longleftrightarrow$ vertex

1-dim face $\longleftrightarrow$ edge

2-dim face $\longleftrightarrow$ triangle

k-dim face $\longleftrightarrow$ k simplex

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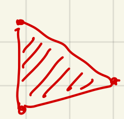

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$\mathcal{F}(\Delta) = \{ F \in \Delta \mid F \text{ maximal w.r.t. } \subseteq \}$ $\longrightarrow$ facets = maximal faces

$I(\Delta) = \langle x_F \mid F \in \mathcal{F}(\Delta) \rangle$ where $x_F = \prod_{\{i_1, \dots, i_k\} \in F} x_{i_1} \cdots x_{i_k}$

facet ideal of Δ

e.g. $F = \{1,2,3\} \longrightarrow x_1 x_2 x_3 = x_F$

$F = \{1,4\} \longrightarrow x_1 x_4 = x_F$

Special case: all generators have degree 2

$[n] = \{1 \ldots n\} \longleftrightarrow$ 0-dim faces $\longleftrightarrow x_i$

edges $\{i,j\} \longleftrightarrow$ 1-dim faces $\longleftrightarrow x_i x_j$

The simplicial complex is a simple graph G

$I(G) =$ edge ideal.

Remark: The combinatorics of simple graphs is much simpler than that of higher dimensional simplicial complexes.

For this reason, very often problems about squarefree monomial ideals are solved for edge ideals and (wide) open for squarefree monomial ideals with generators in higher degrees!

We will see some examples next time.

Stanley - Reisner Theory

Δ simplicial complex

$$\mathcal{N}(\Delta) = \{ F \in \underbrace{\mathcal{P}([n]) \setminus \Delta}_{\text{non faces}} \mid F \text{ minimal w.r.t. } \subseteq \} \to \text{minimal non-faces}$$

$$I_\Delta = \langle x_F \mid F \in \mathcal{N}(\Delta) \rangle$$ Stanley-Reisner ideal

$$\left\{ \begin{array}{c} \text{squarefree monomial} \\ \text{ideals} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{facet ideals } I(\Delta) \\ \text{of } \Delta \end{array} \right\}$$

$$\updownarrow$$

$$\left\{ \begin{array}{c} \text{Stanley-Reisner ideals} \\ \text{of } I_{\Delta^\vee} \end{array} \right\}$$

$\Delta^\vee$: Alexander dual of Δ

$$\mathcal{F}(\Delta^\vee) = \{ \underbrace{[n] \setminus F}_{\text{complement of the face } F} : F \in \mathcal{N}(\Delta) \}$$

Hochster's Formula (1977): Let Δ be a simplicial complex on $[n]$. For a monomial u in $k[x_1, \dots, x_n]$ let $U = \{ j \in [n] : x_j \mid u \}$ and let Δ_U be the restriction of Δ to U.

Then, for all $i, j \geq 0$

$$\beta_{ij}(I_\Delta) = \sum_{\substack{u \text{ squarefree} \\ \text{monomial, } \deg u = j}} \dim_k \underbrace{\tilde{H}^{j-i-2}}_{\text{reduced simplicial cohomology}} (\Delta_U, k).$$

Simpler formula by Eagon - Reiner (1998)

"Easier" simplicial cohomology considering $\Delta^\vee$ instead of Δ.