Combinatorial Commutative Algebra - II

11/06/20

Outline:

I. Stanley-Reisner correspondence as a dictionary

- combinatorial characterization of linear resolutions / CM property
  (in terms of combinatorial properties of simplicial complex / graph)
- Rees algebra

II. Initial ideal $in_< I$ & polarization

- behavior of graded Betti numbers and projdim under passage to $in_<(I)$
- behavior of graded Betti numbers and projdim under polarization

Last time: Δ simplicial complex on $[n] = \{1, \dots, n\}$

$\mathcal{F}(\Delta) = \{F \in \Delta \mid F \text{ maximal w.r.t. } \subseteq\}$ $\longrightarrow$ maximal faces = Facets

$\mathcal{N}(\Delta) = \{F \in \mathcal{P}([n]) \setminus \Delta \mid F \text{ minimal w.r.t. } \subseteq\}$ $\longrightarrow$ minimal non-faces

For $F \in \Delta$ let $x_F = \prod_{\{i_1, \dots, i_k\} \in F} x_{i_1} \cdots x_{i_k}$ $\longrightarrow$ monomial in $k[x_1, \dots, x_n]$

vertices of Δ $\updownarrow$ variables

$I(\Delta) = \langle x_F \mid F \in \mathcal{F}(\Delta) \rangle$ facet ideal

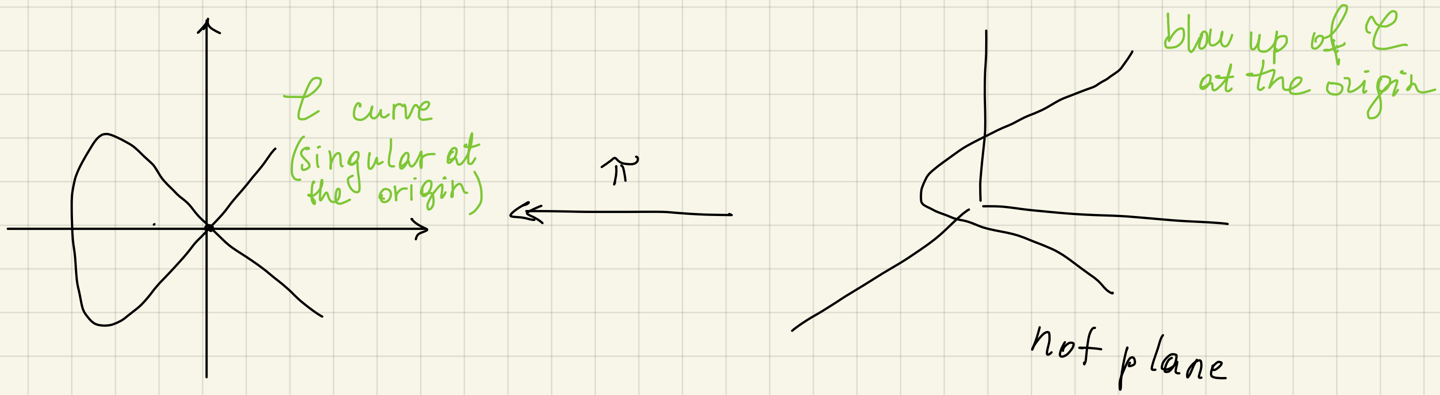
$I_\Delta = \langle x_F \mid F \in \mathcal{N}(\Delta) \rangle$ Stanley-Reisner ideal

$$\left\{\begin{array}{c}\text{facet ideals} \\ I(\Delta) \text{ of } \Delta\end{array}\right\} \xleftrightarrow{1-1} \left\{\begin{array}{c}\text{squarefree monomial} \\ \text{ideals in } k[x_1, \dots, x_n]\end{array}\right\} \xleftrightarrow{1-1} \left\{\begin{array}{c}\text{Stanley-Reisner ideals} \\ I_{\Delta^\vee} \text{ of } \Delta\end{array}\right\}$$

$\Delta^\vee$: Alexander dual of Δ

① When is R/I Cohen-Macaulay?

② What are the equations defining the blow up along I?



A singular curve can be thought of as the projection of a curve on a larger space, where the projection map glues together two branches of the function at the origin.

Cohen-Macaulay property of squarefree monomial ideals

Recall: $R = k[x_1, \dots, x_n]$ $\quad I \subseteq R$ ideal

R/I is Cohen-Macaulay if there exists a regular sequence of length $n - \mathrm{codim}(I)$

Theorem [Eagon-Reiner]: Δ simplicial complex

$R/I_{\Delta^\vee}$ is CM if and only if I_Δ has a linear resolution.

Def: $I = (f_1, \dots, f_s)$ homogeneous ideal, $\deg f_i = d \ \forall i$

I is said to have a linear resolution if $\beta_{ij} = 0$ for all $j \neq i + d$ $\longrightarrow$ so, the Betti table has a lot of 0's

Proof relies on Hochster formula for Betti numbers (simplicial cohomology) $\longrightarrow$
- difficult to compute
- depends on char(k).

Question: Can these properties be understood directly from the combinatorics of Δ?

Def:
- A simplicial complex Δ is pure if all of its facets have the same dimension.
- A simplicial complex Δ is shellable if it is pure and there exists an ordering of its facets $F_1, \dots, F_s$ so that the set $\{G \subseteq F_t \mid G \nsubseteq F_\ell \ \forall \ell < t\}$ contains a unique minimal element.

Fact: Whenever Δ shellable, then R/I_Δ is CM.

Rees algebras of squarefree monomial ideals

$R = k[X_1, \dots, X_n]$, $I = (u_1, \dots, u_m)$. The Rees algebra of I

is $\mathcal{R}(I) = R[It] = R \oplus It \oplus I^2t^2 \oplus \cdots \subseteq R[t]$ → subalgebra

$$I^2 = \langle fg \mid f, g \in I \rangle$$

$$I^j = \langle f_{i_1} \cdots f_{i_j} \mid f_{i_k} \in I \text{ for all } 1 \leq j \leq k \}$$

When $I = (u_1, u_2, \dots, u_m)$ is a monomial ideal then

$$I^j = \langle u_{i_1} \cdots u_{i_j} \mid u_{i_k} \text{ is a generator of } I \rangle$$

Notice: $\mathcal{R}(I) = R[T_1, \dots, T_m] / \mathcal{J}$ → defining ideal of $\mathcal{R}(I)$

denote this with S.

(its generators correspond to the equations defining the blowup)

[Taylor, 1968]: $I = (u_1, \dots, u_m)$ monomial ideal, assume that $\{u_1, \dots, u_m\}$ is a minimal monomial generating set.

For each $s \in \{1, \dots, m\}$, let $\mathcal{I}_s = \{ \alpha = (i_1, \dots, i_s) \mid 1 \leq i_1 \leq \dots \leq i_s \leq n \}$

For $\alpha \in \mathcal{I}_s$, let $u_\alpha = u_{i_1} \cdots u_{i_s}$ and $T_\alpha = T_{i_1} \cdots T_{i_s}$

$$J_s = \Big\{ \underbrace{\frac{u_\beta}{\gcd(u_\alpha, u_\beta)} T_\alpha - \frac{u_\alpha}{\gcd(u_\alpha, u_\beta)} T_\beta}_{\text{follows from Taylor resolution}} \;\Big|\; \alpha, \beta \in \mathcal{I}_s \Big\}$$

Then, the defining ideal of $\mathcal{R}(I)$ is

$$\boxed{\mathcal{J} = \underbrace{SJ_1}_{\text{generators of } T\text{-degree } 1} + \underbrace{S\Big(\bigcup_{i=2}^{\infty} J_i\Big)}_{\text{generators of } T\text{-degree} \geq 2}}$$

generators of T-degree ≥ 2 (this is a finite union in fact)

[Villarreal, 1995]: Let $I = I(G)$ be the edge ideal of a simple graph G.
(= squarefree monomial ideal, with generators of degree 2)

Then, the defining ideal of $\mathcal{R}(I)$ is

$$\mathcal{J} = S J_1 + S\left(\bigcup_{s=2}^{\infty} P_s\right)$$

where

$$P_s = \Big\{ \underbrace{T_\alpha - T_\beta}_{\text{T-degree } 1,\ \text{X-degree } 0} \;\Big|\; u_\alpha = u_\beta \text{ for some } \alpha, \beta \in \mathcal{I}_s \Big\} \longrightarrow \text{toric ideal}$$

Moreover:

- $\mathcal{J} = S J_1$ (an ideal of linear forms) if and only if G is a tree or has a unique cycle of odd length
- The generators of $\bigcup_{s=2}^{\infty} P_s$ correspond to even closed walks in G

For squarefree monomial ideals with gen. of degree ≥ 3 there is no known combinatorial description of defining ideal of Rees algebra.

Constructing monomial ideals and squarefree monomial ideals

$S = k[x_1, \dots, x_n]$ $\quad$ $Mon(S) = \{\text{all monomials in } S\}$

Def: A monomial order on S is a total order $<$ on $Mon(S)$ so that:

(i) $1 < u$ for all $u \in Mon(S)$ with $u \neq 1$

(ii) if $u, v \in Mon(S)$ and $u < v$, then $uw < vw$ for all $w \in Mon(S)$

E.g. Lexicographic order

Let $\underline{x}^{\underline{a}} = x_1^{a_1} \cdots x_n^{a_n}$. We say that $\underline{x}^{\underline{a}} \leq_{lex} \underline{x}^{\underline{b}}$ if either

(i) $\sum_{i=1}^{n} a_i < \sum_{i=1}^{n} b_i$, or

(ii) $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ and the leftmost non-zero component of $\underline{a} - \underline{b}$ is negative

e.g. $u = x_1^2 x_2^3$, $v = x_1 x_2^2 x_3^2 \longrightarrow v \leq_{lex} u$.

Remark: Every polynomial can be written as sum of monomials

$f = a_1 u_1 + \dots + a_s u_s$, $a_i \in K$

- $in_<(f) :=$ monomial u_i which is biggest wrt $<$ — initial monomial of f wrt $<$
- Let I be a homogeneous ideal

$in_<(I) := \langle$ ideal generated by $in_<(f) \mid f$ is in $I \rangle$ initial ideal of I wrt $<$

This is a monomial ideal.

Remark: In general, if $I = (f_1, \dots, f_m)$ homogeneous ideal

$$in_<(I) \neq \langle in_<(f_1), \dots, in_<(f_m) \rangle$$

If $=$ holds, $\{f_1, \dots, f_m\}$ are called a Gröbner basis.

So, from a given homogeneous ideal we can always construct a monomial ideal. Also, from a monomial ideal we can construct a squarefree monomial ideal as follows.

Let $S = k[x_1, \dots, x_n]$, $I = (u_1, \dots, u_m)$ monomial ideal.

$\hookrightarrow$ minimal monomial generators

Write $u_i = \prod_{j=1}^{n} x_j^{a_{ij}}$ for $i = 1, \dots, m$ and suppose $a_{ij} > 1$ for some i, j

$\downarrow$ i.e. I not squarefree

Let $T = S[y]$ and let $J = (v_1, \dots, v_m)$

where
$$v_i := \begin{cases} u_i & \text{if } a_{ij} = 0 \\ \frac{u_i}{x_j} y & \text{if } a_{ij} \geq 1 \end{cases}$$

That is: Since $x_j^2 \mid x_j^{a_{ij}}$, we trade an x_j with a new variable y

Then:
$$\frac{T/J}{(y - x_j)\, T/J} \cong S/I$$
as rings and $y - x_j$ is a non-zerodivisor modulo J

Repeat if necessary $\longrightarrow$ eventually obtain a squarefree monomial ideal J

This technique is called polarization.

Construction of initial ideals and polarization keep track of relevant properties:

Theorem 1: $I \subseteq S$ homogeneous ideal, $<$ a monomial order on S

- $\beta_{ij}(I) \leq \beta_{ij}(\mathrm{in}_<(I))$
- $\mathrm{projdim}(S/I) \leq \mathrm{projdim}(S/\mathrm{in}_<(I))$

Sometimes these are $=$ but most often they aren't

Theorem 2: $I \subseteq S$ monomial ideal, $J \subseteq T$ its polarization

- $\beta_{ij}(I) = \beta_{ij}(J)$
- $\mathrm{projdim}(S/I) = \mathrm{projdim}(T/J)$