Betti numbers and free resolutions

UCR CA Seminar 16/10/2020

R (noetherian) ring often $R = k[x_1, \dots, x_d]$, k field

M fg R-module

How would we describe M?

M fg R-mod $\Rightarrow$ $f_1, \dots, f_n \in M$ generate M

Every element in M is of the form

$$r_1 f_1 + \dots + r_n f_n \qquad r_i \in R$$

so there is a map
$$\begin{aligned} R^n &\xrightarrow{\pi} M \\ (r_1, \dots, r_n) &\longmapsto r_1 f_1 + \dots + r_n f_n \end{aligned}$$

$\mu(M) = n$ = minimal number of generators

the map π is a (minimal) presentation of M

One of two things will happen:

1) M is <u>free</u> $\Leftrightarrow$ $M \cong R^n$ $\Leftrightarrow$ π is an iso

2) M is no free $\Leftrightarrow$ $\ker \pi \neq 0$

Case 2) is more common (and more interesting)

Elements of $\ker \pi$ $\Longleftrightarrow$ relation among the generators of M

$$r_1 f_1 + \cdots + r_n f_n = 0$$

<u>Next</u> to continue describing M, describe $\ker \pi$

- How many relations are there?
- What are the relations among those relations?

$\vdots$

keep going until there are no relations $\Longleftrightarrow$ find a free module

(side note: $A \xrightarrow{f} B \xrightarrow{g} C$ is <u>exact</u> iff $\operatorname{im} f = \ker g$)

$$\begin{tikzcd}
\cdots \arrow[r] & R^{\ell} \arrow[rrr, dashed] \arrow[dr] & & & R^{m} \arrow[rr, dashed, "\pi_1"] \arrow[dr] & & R^{n} \arrow[r, "\pi"] & M \arrow[r] & 0 \\
& & \ker \pi_1 \arrow[urr] \arrow[dr] & & & \ker \pi \arrow[ur] \arrow[dr] & & & \\
& 0 \arrow[ur] & & 0 & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

A free resolution of M is an exact sequence

$$\cdots \to R^{n_p} \longrightarrow \cdots \longrightarrow R^{n_1} \to R^{n_0} \; (\longrightarrow M \to 0)$$

the free resolution of M is <u>minimal</u> if each n_i is smallest possible

$\beta_i(M) := n_i$ in a minimal free resolution (Betti numbers of M)

the <u>projective dimension</u> of M is the length of a minimal resolution of M

<u>Example</u> $\quad I = (xy, xz, yz) \subseteq k[x,y,z] = R.$

$$0 \longrightarrow \overset{1}{R^2} \xrightarrow[\begin{pmatrix} -z & 0 \\ y & y \\ 0 & -x \end{pmatrix}]{\hspace{6em}} \overset{0}{R^3} \xrightarrow[(xy \;\; xz \;\; yz)]{\hspace{6em}} I$$

$$-z\cdot(xy) + y\cdot(xz) = 0$$

$$\operatorname{pdim}(I) = 1$$

$$\beta_0(I) = \mu(I) = 3 \qquad \beta_1(I) = 2 \qquad \beta_{\geq 2}(I) = 0$$

<u>Example</u> $\quad M = R/I \quad$ (a cyclic module $\Leftrightarrow \mu(M) = 1$)

$$0 \longrightarrow R(-3)^2 \xrightarrow[\begin{pmatrix} z & 0 \\ -y & y \\ 0 & -x \end{pmatrix}]{\hspace{4em}} R(-2)^3 \xrightarrow[(xy \;\; xz \;\; yz)]{\hspace{6em}} R \longrightarrow M \longrightarrow 0$$

all of degree 2

all of deg 1

β_{ij} = (i,j)th betti number, number of i-relations of degree j

$R(-a) = R$, but 1 lives in degree a

$\beta_{00} = 1 \qquad \beta_{12} = 3 \qquad \beta_{23} = 2 \qquad \beta_{ij} = 0$ otherwise

Warning A fg R-module could have infinite proj dim!

Example $R = k[x]/(x^3)$ $M = k[x]/(x^2) = R/(x^2)$ cyclic!

$$\cdots \longrightarrow R \xrightarrow{\cdot x^2} R \xrightarrow{\cdot x} R \xrightarrow{\cdot x^2} R \xrightarrow{\pi} M \to 0$$

$\operatorname{pdim} M = \infty \qquad \beta_1(M) = 1$

Theorem (Hilbert's Syzygy theorem) k field

Every fg R-module over $R = k[x_1, \dots, x_d]$ has projdim $\leq d$.

Example $f \in R = k[x_1, \dots, x_d]$, $f \neq 0$, $M = R/(f)$

$$0 \to R \xrightarrow{\cdot f} R \to R/(f) \to 0$$

$$f \cdot g = 0 \underset{\text{domain}}{\overset{R \text{ is a}}{\Longrightarrow}} g = 0$$

What mattered here was that f is regular. $(fg = 0 \Rightarrow g = 0)$

A regular element in any ring R always has resolution

$$0 \to R \xrightarrow{\cdot f} R \to R/f \to 0$$

$f_1, \dots, f_n$ is a regular sequence $\Leftrightarrow (f_1, \dots, f_n)$ is a complete intersection

- f_1 is regular $(fg = 0 \Rightarrow g = 0)$
- f_2 is regular on $R/(f_1)$

$$\left(\begin{array}{ccc} f_2 g \text{ in } R/(f_1) & & f_2 g \in (f_1) \\ \Downarrow & \Leftrightarrow & \Downarrow \\ g = 0 \text{ in } R/(f_1) & & g \in (f_1) \end{array} \right)$$

- f_3 is regular on $R/(f_1, f_2, f_3)$

$\vdots$

- f_n is regular on $R/(f_1, \dots, f_{n-1})$

<u>Sidenote</u> How to think about complete intersections?

In general, an ideal $I = (f_1, \dots, f_n)$ $(\text{in } R = k[x_1, \dots, x_d])$
has height / codimension $h \leq n = \mu(I)$

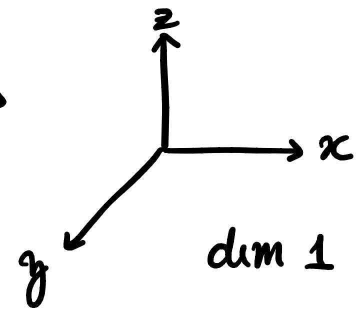
What does that mean?

$$(f_1, \dots, f_n) \longleftrightarrow \begin{cases} f_1 = 0 \\ \vdots \\ f_n = 0 \end{cases} \longleftrightarrow$$ some picture in k^d given by solutions to this system

$$\begin{array}{l} \operatorname{codim}(f_1, \dots, f_n) = d - \text{dimension of the solution set} \\ \qquad \| \\ \operatorname{height}(f_1, \dots, f_n) = \text{defined in terms of chains of primes} \end{array}$$

Example $I = (xy, xz, yz) \subseteq \mathbb{C}[x,y,z]$

$$\begin{cases} xy = 0 \\ yz = 0 \\ xz = 0 \end{cases} \iff \begin{matrix} x=0 \text{ and } z=0 \\ \text{or} \\ x=0 \text{ and } y=0 \\ \text{or} \\ y=0 \text{ and } z=0 \end{matrix} \iff$$



dim 1

height $=$ Codim $= 3 - 1 = 2 \leq 3$

Krull's Height theorem height $(I) \leq \mu(I)$

Fact codim$(f_1, \dots, f_n) = \mu(f_1, \dots, f_n) \iff f_1, \dots, f_n$ is a regular sequence

Equivalently, the minimal free resolution of $R/(f_1, \dots, f_n)$ is the Koszul complex

$$0 \to R^n \longrightarrow \cdots \longrightarrow R^{\binom{n}{i}} \longrightarrow \cdots \longrightarrow R^n \xrightarrow[(f_1, \dots, f_n)]{} R$$
$$i$$

the maps are also easy to describe (up to signs)

Example $0 \to R \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} 0 & f_3 & -f_2 \\ -f_3 & 0 & f_1 \\ f_2 & -f_1 & 0 \end{pmatrix}} R^3 \xrightarrow{(f_1 \ f_2 \ f_3)} R \to 0$

Moral I is a complete intersection $\iff$ the resolution of R/I is the simplest possible