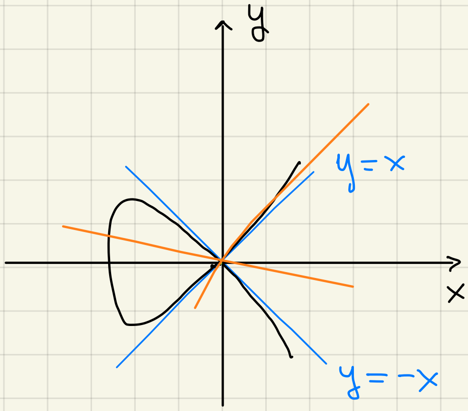
Rees algebras and multiplicity

02/12/21

Curve singularities and multiplicity



$\mathcal{C} = \{(x,y) \in \mathbb{A}^2_k \mid y^2 - x^2(x+1) = 0\}$

plane curve, singular at $(0,0)$

$$R = \mathcal{O}_{\mathcal{C},(0,0)} = \left(k[X,Y] \big/ (Y^2 - X^2(X+1)) \right)_{(X,Y)}$$

local ring

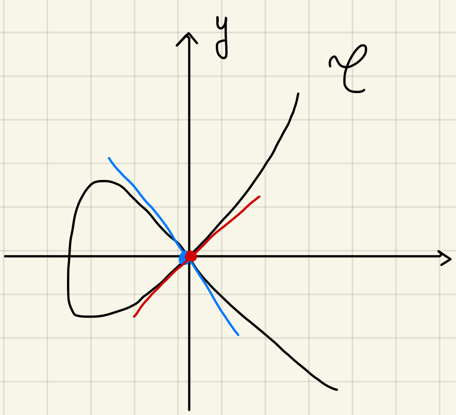
The tangent lines at $(0,0)$ limit the positions of the secant lines through the origin

Tangent cone $\longrightarrow$ $k[X,Y] \big/ (Y^2 - X^2) = \mathrm{gr}_{(x,y)}(R)$

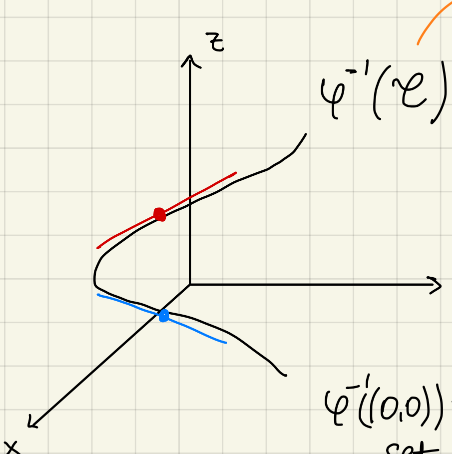
associated graded ring of R w.r.t. (x,y)

tangent lines at $(0,0)$ = # intersection pts of secant lines through $(0,0)$ = degree $(Y^2 - X^2)$ = 2 = multiplicity at $(0,0)$

If you have a non singular curve, multiplicity is 1



$\xleftarrow{\ \varphi\ }$

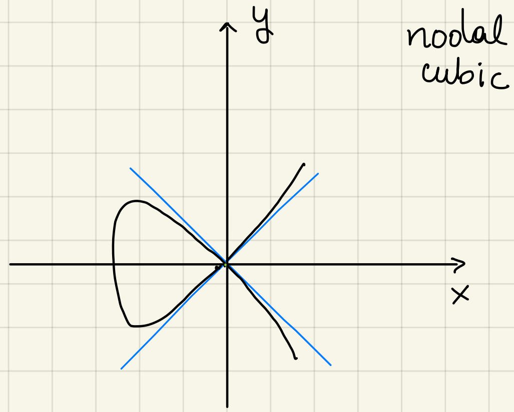


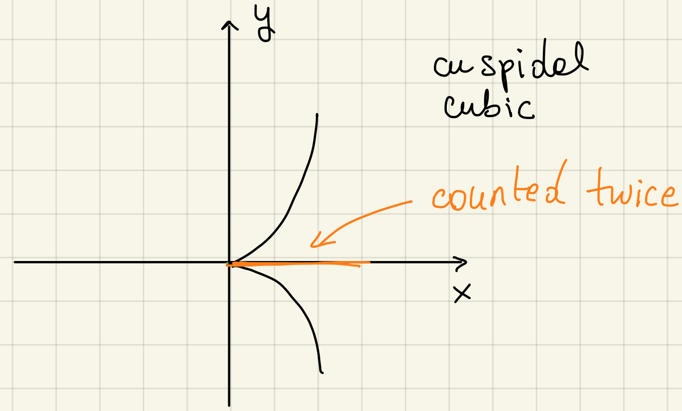
homog. coord. ring of $\mathcal{R}(x,y)$

$\varphi^{-1}(\mathcal{C}) :=$ blow up of $\mathcal{C}$ at the origin

$\varphi^{-1}((0,0)) =$ exceptional set of the blowup

homog. coord. ring of $\mathrm{gr}_{(x,y)}(R)$.





$$\mathcal{C} = \{(x,y)\in \mathbb{A}^2_k \mid y^2 - x^2(x+1) = 0\} \qquad\qquad \mathcal{C} = \{(x,y)\in \mathbb{A}^2_k \mid y^2 - x^3 = 0\}$$

$$gr_{(x,y)}(R) = k[X,Y]/(X^2-Y^2) \qquad\qquad gr_{(x,y)}(R) = k[X,Y]/(Y^2)$$

multiplicity = 2 $\qquad\qquad$ multiplicity = 2

Caution: multiplicity captures the presence of a singularity but alone it doesn't distinguish different kinds of singularities

only one maximal ideal

R Noetherian local ring, dimension of R = d, $I = (f_1, \dots, f_n)$

$$\mathcal{R}(I) = \bigoplus_{j\geq 0} I^j \cong \bigoplus_{j\geq 0} I^j t^j = R[It] = R[f_1 t, \dots, f_n t] \subseteq R[t]$$

$$gr_I(R) \cong \mathcal{R}(I)\big/ I\mathcal{R}(I) = \bigoplus_{j\geq 0} I^j \Big/ I\Big(\bigoplus_{j\geq 0} I^j\Big) = \bigoplus_{j\geq 0} I^j / I^{j+1}$$

Hilbert multiplicity of $gr_I(R)$ $\qquad e(gr_I(R)) = \lim_{j\to\infty} \dfrac{\lambda(I^j/I^{j+1})}{j^{d-1}}(d-1)!$

// and coincide with the multiplicity

Hilbert multiplicity of I $\qquad e_I(R) = \lim_{j\to\infty} \dfrac{\lambda(R/I^j)}{j^d} d!$

λ = length

Remark: these lengths are not always finite!

I a power of m, or a primary $\to$ this is well-defined

Cohen-Macaulay rings

<u>Recall</u>: R local ring with unique maximal ideal m

$\{x_1, \dots, x_t\}$ is a <u>regular sequence</u> in R if

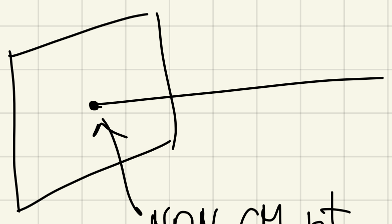
- none of $x_1, \dots, x_t$ is a unit
- x_1 is a non-zerodivisor on R
- x_2 is a non-zerodivisor on $R/(x_1)$

  ⋮
- x_t is a non-zerodivisor on $R/(x_1, \dots, x_{t-1})$

R Noetherian local ring, then the maximal length of a regular sequence in R is $\leq \dim R$.

<u>Def</u>: R is Cohen-Macaulay (CM) if and only if $\dim R =$ max. length of a regular sequence.

- Equidimensional



- locally connected in codim 1.
- "Nice" sheaf cohomology (interesting homological characterization)

Natural questions

- How do CM singularities behave under blow-up constructions?
- What role does the Hilbert-Samuel multiplicity have in understanding CM singularities?

Abhyankar's Inequality (1967)

R CM local ring with maximal ideal m and dimension d.

$$e(m) \geq \mu(m) - d + 1$$

$\mu(m)$: min. number of gen. = dim. of tangent space at (0,0)

Notice:
non-singular curve
R $\dim R = 1$
$\mu(m) = 1$
$e(m) = 1$
minimal multiplicity.

- [Sally, 1977–1983]: R CM local ring with maximal ideal m, $\dim R = d$
  - If m has minimal multiplicity $e(m) = \mu(m) - d + 1$, then $gr_m(R)$ is CM
  - If m has almost minimal multiplicity $e(m) = \mu(m) - d + 2$, then $gr_m(R)$ is almost CM $\longrightarrow$ length of max. reg. seq. $\leq d-1$

Conjecture: How far $gr_m(R)$ is from being CM is encoded in its Hilbert polynomial

$\longrightarrow$ $e(gr_m(R)) = e(m)$ leading coefficient of Hilbert polynomial

almost almost minimal $\longrightarrow$ $\mu(m) - d + 3 = e(m)$

Sally's conj. is not solved in this case.

minimal multiplicity $\iff$ $\exists\, a_1, \dots, a_d \in m$ so that $m(a_1, \dots, a_d) = m^2$ and the images of $a_1, \dots, a_d$ in $gr_m(R)$ form a regular seq.

The case of $\mathfrak{m}$-primary ideals

R CM local ring with maximal ideal $\mathfrak{m}$, $\dim R = d$

I $\mathfrak{m}$-primary ideal ⟶ examples: powers $\mathfrak{m}^s$, λ function is well-defined

- [Valla, 1979]: $e(I) \geq \lambda(I/I^2) - (d-1)\,\lambda(R/I)$

  If I has minimal multiplicity then $gr_I(R)$ is CM

- [Rossi-Valla, 1996 ; Rossi, 2000]

  If I has almost minimal multiplicity then $gr_I(R)$ is almost CM ($t \leq \dim - 1$) and Sally's conjecture holds

  ⟶ Ratliff-Rush filtration

Notice: $\lambda\left(\mathfrak{m}/\mathfrak{m}^2\right) = \mu(\mathfrak{m})$

When you blow up a CM singularity (point / point with multiplicity), the exceptional set of the blow up is still CM

--- as long as you started with minimal multiplicity!

Theorem: [Huneke 1982, Trung-Ikeda 1989]

R CM local ring, I an ideal containing a non-zerodivisor. Then:

- $\mathcal{R}(I)$ CM $\Longrightarrow$ $gr_I(R)$ CM
- $gr_I(R)$ CM $+$ $a(gr_I(R)) < 0$ $\Longrightarrow$ $\mathcal{R}(I)$ CM

  ⟶ a-invariant (numerical invariant defined using local cohomology)

Application: defining ideal of Rees algebras

- [Morey-Ulrich, 1996]: $R = k[x_1, .., x_d]$, k an infinite field

  $I = (f_1, \dots, f_n)$ R-ideal of codimension 2 with a <u>linear</u> presentation
  $$0 \to R^{n-1} \xrightarrow{A} R^n \to I \to 0.$$
  Suppose that for all $i \leq d-1$ $\operatorname{ht} I_{n-i}(A) \geq i+1$.

  Then, $\mathcal{J} = \mathcal{L} + I_d(B)$, where B is the Jacobian dual of A.
  $$\Updownarrow$$
  $\mathcal{R}(I)$ is CM and $I_{n-d}(A) = I_1(A)^{n-d}$

- [Boswell-Mukundan, 2016]: $R = k[x_1, .., x_d]$, k an infinite field

  $I = (f_1, \dots, f_n)$ an R-ideal of codimension 2, with a presentation
  $0 \to R^{n-1} \xrightarrow{A} R^n \to I \to 0$, where the entries of A are
  <u>linear except one column</u> of degree $m \geq 1$.

  Suppose that for all $i \leq d-1$ $\operatorname{ht} I_{n-i}(A) \geq i+1$ and $\mu(I) = d+1$.

  Then, $\mathcal{J} = \mathcal{L} + I_d(B_m)$ (B_m = m^{th} iterated Jacobian dual of A)

  Moreover, $\mathcal{R}(I)$ is almost CM, and not CM unless $m=1$.