What is Commutative Algebra and What will this seminar be about?
9/10/2020

Commutative Algebra has connections to:

- Homological Algebra
- Algebraic Geometry
- Number theory
- Arithmetic Geometry → p-derivations, perfectoid spaces
- Combinatorics → combinatorial commutative algebra
- Invariant theory → rings of invariants
- Representation theory
- Differential algebra (differential operators, rings of $\mathcal{D}$-modules)
- Topology, especially rational homotopy theory
- Lie Algebras
- Cluster algebras

So what do commutative algebraists study?

- <u>Rings</u>, where of course all rings are commutative, with $1 \neq 0$

and

- <u>Modules</u> over such rings, especially finitely generated R-modules

(modules are to rings what vector spaces are to fields
A module is an abelian group ⊕ a compatible action of R on M)

Some subfields/topics: (disclaimer: these are just some of my favorites)

- <u>Homological Algebra</u>

Roughly: things involving complexes (of R-modules)

this includes:

- Categorical ideas: $\operatorname{mod} R :=$ category of fg R-modules
  $\mathcal{D}(R) :=$ derived category of complexes of R-modules

- Behavior of Ext and Tor $\quad \operatorname{Ext}^i_R(M,N)$, $\operatorname{Tor}^R_i(M,N)$

- Local cohomology $\quad$ • Injective resolutions

- More concrete things: free/projective resolutions, betti numbers

- Homological Conjectures (some still open)

  - Homological techniques appear all over Commutative Algebra <u>often</u>.

- <u>Combinatorial Commutative algebra</u> $\quad$ there is a dictionary

$$\begin{array}{ccc} \text{simplicial complexes} & \longleftrightarrow & \text{Stanley-Reisner rings} \\ \text{combinatorial properties} & \longleftrightarrow & \text{ring-theoretic properties} \\ & \cup\!\text{I} & \\ \text{simple graphs} & \longleftrightarrow & \text{"edge ideals"} \end{array}$$

- <u>characteristic</u> p <u>commutative algebra</u> (eg, $\mathbb{Z}/p[x_1, \dots, x_d]/I$)

Rings of prime characteristic come with a powerful tool: Frobenius

$$p = 0 \text{ in } R \Rightarrow (x+y)^p = x^p + y^p \rightsquigarrow x \mapsto x^p \text{ homomorphism}$$

properties of Frobenius $\Leftrightarrow$ ring theoretic properties of R $\Leftrightarrow$ (geometric) singularities

Connections to char 0 singularities, Minimal Model Program

Allows to answer char 0 questions sometimes, by Reduction to char p

(sometimes)
solving a problem in char p $\Longrightarrow$ solving the problem over any field

- <u>Computational Commutative algebra</u>

  - How do we compute actual examples?
  - Algebra software like Macaulay2, CoCoA, Singular, etc.

- <u>other classical topics</u> :

various notions of powers, Rees algebras, integral closure, dimension theory

Typical rings we study:

- $\dfrac{k[x_1, \dots, x_d]}{I}$, k field, I ideal (polynomial rings)

- $\dfrac{k[[x_1, \dots, x_d]]}{I}$, k field, I ideal (power series rings)

- local Rings $(R, \mathfrak{m})$
  $\hookrightarrow$ unique maximal ideal

can be obtained by localization:
you take a ring R (not necessarily local) and a prime ideal P, and zoom in on P, turning P into the unique maximal ideal and inverting everything outside of P. New ring R_P

eg: $R = \mathbb{Z}$, $P = (0) \Rightarrow R_P \cong \mathbb{Q}$

other examples: $\mathbb{Z}$, $\mathbb{Z}_p$ (p-adics), constructions over these

Technical assumptions we will see a lot:

- Often rings are noetherian

all ideals are fg $\Leftrightarrow$ all ascending chains of prime ideals stabilize

Examples

- Fields
- Polynomial Rings over fields (in finitely many vars)
- Quotients and localizations of noetherian rings

Hilbert's Basis theorem R noetherian $\Rightarrow R[x]$ noetherian

(every system of polynomial equations in finitely many variables can be reduced to finitely many equations)

- Conclusion: rings essentially of finite type over a field

$$\left(\frac{k[x_1, \dots, x_d]}{I} \right)_P$$

- Often rings are domains (no zero divisors)
- Sometimes rings are graded

baby example: $k[x_1, \dots, x_d]$, $\deg x_i = 1$

More advanced example: $\dfrac{k[x_1, \dots, x_d]}{\text{homogeneous } I}$

$$f \text{ homogeneous of degree } d \Leftrightarrow f(\lambda \underline{x}) = \lambda^d f(\underline{x})$$

- In some sense, all rings are quotients of a regular ring

Roughly speaking, $k[x_1, \dots, x_d]$ or $k[[x_1, \dots, x_d]]$

fields $\longleftarrow$ boring $\qquad k$

$\cap$|

regular rings $\longleftarrow$ nice and beautiful $\qquad k[x,y,z]$

$\cap$|

complete intersections $\longleftarrow$ as close to regular as possible

$k[x,y,z]/(x^2,y^3,z^4)$

$\cap$|

Gorenstein rings $\longleftarrow$ good homological properties

$k[x,y,z]/(x^2,y^2,xz,yz,z^2-xy)$

$\cap$|

Cohen-Macaulay rings $\longleftarrow$ just nice enough $\qquad \dfrac{k[x,y,z]}{(x^2,y^2,xy)}$

$\cap$|

all sorts of wild stuff $\qquad k[x,y,z]/(xy,xz)$

"Life is worth living in a Cohen-Macaulay ring"
— Mel Hochster

Characteristic

- R contains a field (equicharacteristic)

typical example: $\dfrac{k[\underline{x}]}{I}$, $\left(\dfrac{k[\underline{x}]}{I}\right)_P$

this implies one of two (mutually exclusive) things:

① R has prime characteristic $p>0$, so

$$\underbrace{1+\cdots+1}_{p \text{ times}} = 0$$

and $R \supseteq \mathbb{F}_p \cong \mathbb{Z}/p$.

② R contains $\mathbb{Q}$, and has equicharacteristic 0

$$\underbrace{1 + \cdots + 1}_{\geq 1} \neq 0$$

and all quotients of R have characteristic 0.

or

③ R does not contain a field $\equiv$ R has mixed characteristic

so: R has characteristic 0, meaning

$$\underbrace{1 + \cdots + 1}_{\geq 1} \neq 0$$

but some quotient of R has characteristic p

Examples $\mathbb{Z}$, $\mathbb{Z}_p$, many examples from number theory

Note Many open problems in Commutative Algebra have been solved in equicharacteristic but not mixed char. Recently, there has been an explosion of advances in mixed char stemming in part from André's solution of the Direct Summand Conjecture using perfectoid spaces techniques.