The Friendship between commutative algebra
and rational homotopy theory

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THROUGH THE LOOKING GLASS: A DICTIONARY BETWEEN RATIONAL HOMOTOPY THEORY AND LOCAL ALGEBRA

by
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§ 0. INTRODUCTION

"Now, if you'll only attend, Kitty,... I'll tell you all my ideas about Looking-glass House. First, there's the room you can see through the glass - that's just the same as our drawing room, only the things go the other way."

Alice [C]

Homological methods, originally invented as tools for algebraic topologists, have almost from their inception played an important role in the study of rings. This has led to any number of analogies between the two subjects and to a certain overlap of terminology.

More recently it has developed that if one restricts attention to rational homotopy theory (within topology) and to commutative rings (within algebra) one gets a particularly coherent analogy of unusual scope and power. This has made it possible to use intuition and techniques from topology to prove theorems in algebra, and conversely.

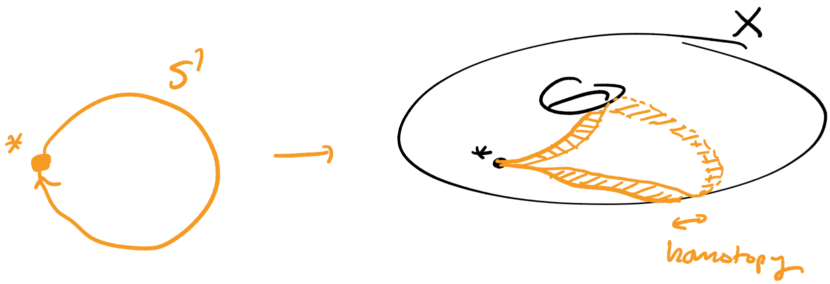
from conference proceedings
"Algebra, Algebraic Topology and their interactions" Stockholm 1983

So this talk is about commutative rings and topological spaces

$\mathbb{Z}$ and $\mathbb{Q}$ "rational"

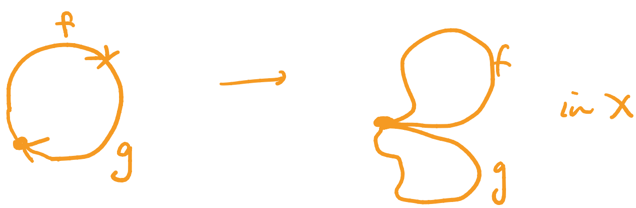
X a based space (i.e. chosen point $* \in X$) & path connected

$$\pi_1(X) = \left\{ \text{maps } (S^1, *) \longrightarrow (X, *) \right\} \Big/ \text{homotopy}$$



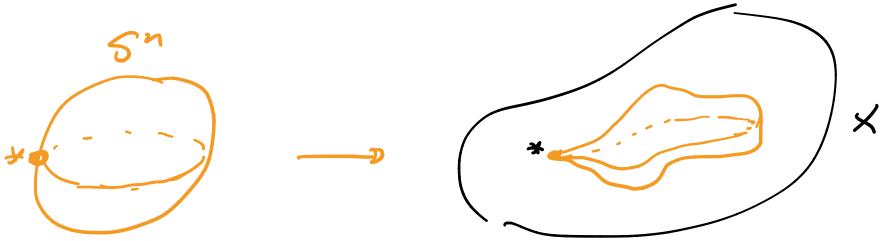
this is a group: $f, g \in \pi_1(X)$

$\Rightarrow$ fg:



But also there are

$$\pi_n(X) = \frac{\{\text{maps } (S^n, *) \longrightarrow (X, *)\}}{\text{homotopy}}$$



Also a group: $f, g \in \pi_n(X)$

$fg : S^n \longrightarrow X$

can rotate sphere like this ...

i.e. pinch equator

... $\pi_n(X)$ is abelian for $n \geq 2$

So sometimes write $f+g$ for this operation instead

Anyway: Topologists want to understand $\pi_n(X)$

eg $\pi_n(S^m) = ?$ very complicated !

Thm (Hopf)

$$\pi_n(S^m) = \begin{cases} 0 & \text{if } n < m \\ \mathbb{Z} & \text{if } n = m \\ \text{can be nontrivial} & \text{if } n > m \text{ !} \end{cases}$$

eg Hopf fibration $S^3 \to S^2 \Rightarrow \pi_3(S^2) = \mathbb{Z}$

Wikipedia: homotopy groups of spheres

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$

very complicated $\longrightarrow \ldots$

it's impossible to say anything general except...

Thm (Serre) all $\pi_n(S^m)$ are finitely generated abelian groups and

$$\operatorname{rank} \pi_n(S^m) = \begin{cases} 1 & \text{if } m=n \quad \text{(Hopf)} \\ 1 & \text{if } m \text{ even and } n=2m-1 \\ 0 & \text{otherwise.} \end{cases}$$

(# of copies of $\mathbb{Z}$)

ie $\pi_n(S^m)$ is finite except along these two lines

(finite part is almost impossible to understand)

another way to describe Serre's thm is

$$\pi_n(S^m) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathbb{Q} & m=n \text{ or } m \text{ even } n=2m-1 \\ 0 & \text{o/w.} \end{cases}$$

Rational homotopy theory (RHT): ignore the finite part

work with $\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ instead

becomes much more computable

Usually restrict attention to simply connected spaces:

$\pi_1(X)=0$ (since π_1 not always abelian and that ruins things)

<u>Note</u>: rationalize an abelian group by $-\otimes_{\mathbb{Z}} \mathbb{Q}$

this can be done geometrically, can rationalize a space by some construction $X \rightsquigarrow X_{\mathbb{Q}}$

and $\pi_n(X_{\mathbb{Q}}) = \pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q}$

(RHT is the study of <u>rational</u> <u>spaces</u> up to homotopy.)

<u>One more thing</u>: X a simply connected based space

$\pi_*(X)_{\mathbb{Q}} = \bigoplus_{n \geq 2} \pi_2(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a graded vector space over $\mathbb{Q}$

<u>Thm</u> (Whitehead/Samuelson/Massey-Uehara)

there another product $[-,-] : \pi_i(X)_{\mathbb{Q}} \times \pi_j(X)_{\mathbb{Q}} \to \pi_{i+j-1}(X)_{\mathbb{Q}}$

this makes $\pi_{*+1}(X)_{\mathbb{Q}}$ into a <u>graded</u> <u>Lie</u> <u>algebra</u>

i.e. $\mathbb{Q}$ linear, anti symmetric, and satisfies Jacobi identity.

Loop spaces: $X \rightsquigarrow \Omega X = \{\text{maps } S^1 \to X\}$

(not upto homotopy)

has a product $\Omega X \times \Omega X \longrightarrow \Omega X$

join loops together

I mention this for <u>two</u> reasons:

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path components of ΩX $\quad \pi_0(\Omega X) \cong \pi_1(X)$

$\vdots$

$\pi_n(\Omega X) \cong \pi_{n+1}(X)$

$*=1 \quad *=2$

<u>EG</u> $X = S^2 \qquad \pi_*(\Omega S^2)_{\mathbb{Q}} = \pi_{*+1}(S^2)_{\mathbb{Q}} = \mathbb{Q}a \oplus \mathbb{Q}b$.

with operation $[a,a] = b$.

Back to commutative algebra:

$$R = \frac{k[[x_1 \dots x_n]]}{f_1 \dots f_c}$$

a local ring, k field
and relations
$f_i \in (x_1 \dots x_n)^2$

R is <u>regular</u> if $c = 0$ $\left(R = k[[x_1 \dots x_c]] \right)$

R is <u>complete intersection</u> if $\dim R = n - c$.

<u>First connection</u> (Roos was interested in this)

$$\dots \to P_1 \to P_0 \to k$$

free resolution,
each $P_i = R^{\beta_i}$ ← betti numbers

$$P_R(t) = \sum_{i \geq 0} \beta_i t^i$$

Poicaré series of R

X nice space (manifold)

⇝ ΩX loop space

homology $H_*(\Omega X; \mathbb{Q})$

$\beta_i = \dim_{\mathbb{Q}} H_i(\Omega X; \mathbb{Q})$

betti numbers of X

$$P_X(t) = \sum_{i \geq 0} \beta_i t^i$$

Poicaré series of X

eg R is regular

$\Rightarrow \quad P_R(t) = \dfrac{1}{(1-t)^n}$

and R is complete int

$\Rightarrow P_R(t) = \dfrac{(1+t^2)^c}{(1-t)^n}$

eg $X = S^m$ $\quad m$ odd

$P_X(t) = \dfrac{1}{1-t^{m-1}}$

eg $X = S^m$ $\quad m$ even

$P_X(t) = \dfrac{1+t^{2m-2}}{1-t^{m-1}}$?

All rational functions (ie $\frac{\text{pol}}{\text{pol}}$)

so Kaplansky/Serre asked

Question: must P_R always be rational?

Same for P_X?

Note: Kaplansky was interested in the number of closed geodesics in X = hyperbolic manifold, with a given length

closed geodesics are loops $\Rightarrow$ live in ΩX

$\Rightarrow$ P_X tells you about this.

Roos (a commutative algebraist) realized these questions were actually connected, not just similar.

Anick (a topologist) proved the answer is no for spaces.

$\Longrightarrow$ the answer is no for rings too.

work of Roos

Rings R s.t. P_R is irrational are called bad rings and the first examples were constructed using topology.

Next connection: Avramov constructed the homotopy Lie algebra $\pi^*(R)$ of a local ring!

This is a graded Lie algebra over k (residue field of R) instead of $\mathbb{Q}$

Always $\pi^*(R) = \pi^1(R) \oplus \pi^2(R) \oplus \pi^3(R) \cdots$

$$\begin{array}{cc} \pi^1(R) & \pi^2(R) \\ \text{\rotatebox{90}{$\cong$}} & \text{\rotatebox{90}{$\cong$}} \\ k^n & k^c \end{array}$$

the numbers $\varepsilon_i(R) = \dim \pi^i(R)$ are the **deviations** of R.

deviations $\Rightarrow$ betti numbers: there is a formula

$$P_R(t) = \frac{\prod_{i \text{ even}} (1+t^i)^{\varepsilon_i}}{\prod_{i \text{ odd}} (1-t^i)^{\varepsilon_i}}$$

Thm (Gulliksen)

R is regular $\iff$ $\varepsilon_2 = 0$ $\iff$ $\varepsilon_{\geq 2} = 0$.

R is complete int $\iff$ $\varepsilon_3 = 0$ $\iff$ $\varepsilon_{\geq 3} = 0$.

From the formula above

you see if $\varepsilon_{>>0} = 0$ then $P_R(t)$ is rational

... but this never happens except for ci's !!

Thm (Félix-Halperin-Thomas[82], Gulliksen[87], Halperin[87], Avramov[99])

(the Elliptic-hyperbolic dychotomy)

Rings $\left[\begin{array}{l} \text{If } R \text{ is complete int then } \pi^*(R) = \pi^1 + \pi^2 \\ \text{If } R \text{ is not comp. int then } \pi^*(R) \text{ grows exp.} \end{array}\right.$

i.e. $\exists\, c > 1$ s.t. $\underbrace{\dim \pi^{\leq i}(R)}_{\varepsilon_1 + \cdots + \varepsilon_i} > c^i$ for all i

"It's unknown if the deviations are eventually increasing."

Spaces X simply con. manifold $\left[\begin{array}{l} \text{either } \exists\, n \text{ s.t. } \pi_{\geq n}(X)_{\mathbb{Q}} = 0 \quad \text{"elliptic spaces"} \\ \text{OR} \quad \pi_*(X) \text{ grows exponentially} \quad \text{"hyperbolic spaces"} \end{array}\right.$

(same sense as above)

Avramov used this result to prove a big conjecture of Quillen about André-Quillen cohomology

Example $\dfrac{k[[x\ y]]}{xy} \Rightarrow \pi^* = \begin{matrix} & 1 & 2 \\ & ak & \\ & bk & \oplus\ ck \end{matrix}$ (Sjödin)

with $[a,b] = c$.

Example $\dfrac{k[x,y]}{(x,y)^2} \Rightarrow \pi^* =$ free Lie algebra on 2 gens degree 1.

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other similarities: if $F \to X \to B$ is a "fibre sequence"

then is a long exact sequence

$$\cdots \to \pi_i(F) \longrightarrow \pi_i(X) \to \pi_i(B) \to \pi_{i-1}(F) \to \cdots$$

if $R \to S$ is a flat map $F = S \otimes_R k$

$\Rightarrow$ $R \to S \to F$ "fibre sequence"

$\Rightarrow$ long exact sequence

$$\cdots \leftarrow \pi^{i+1}(F) \leftarrow \pi^i(R) \leftarrow \pi^i(S) \leftarrow \pi^i(F) \leftarrow \cdots$$

Avramov used $\pi^*(R)$ to solve "Grothendieck's localization problem"

Thm if R is complete intersection and $p \subseteq R$ prime then R_p is complete intersection too.

Finally More hints on the looking glass:
(can't fully explain what these things are)

$$\operatorname{Ext}^*_R(k,k) \longleftrightarrow H_*(\Omega X; \mathbb{Q})$$

$$\beta_i^R = \dim \operatorname{Ext}^i_R(k,k) \longleftrightarrow \beta_i^X = \dim H_i(\Omega X, \mathbb{Q})$$

$$\pi^*(R) \longleftrightarrow \pi_*(\Omega X)_{\mathbb{Q}}$$

long exact sequence (cohomological) $\longleftrightarrow$ long exact sequence (homological)

$$U\pi^*(R) \cong \operatorname{Ext}^*_R(k,k) \longleftrightarrow U\pi_*(X)_{\mathbb{Q}} \cong H_*(\Omega X; \mathbb{Q})$$

Milnor – Moore thm.

Milnor - Moore, Assmus, Levin, Sjoeller
André, Sjödin

$$R \longleftrightarrow H^*(X; \mathbb{Q})$$ singular cohomology