Homological Methods in Commutative Algebra

Problem Set 1

Throughout, all rings all commutative and noetherian, and all modules are finitely generated.

Problem 1. Write the Koszul complex on 3 elements f_1, f_2, f_3.

Problem 2. Let I be a proper nonzero ideal in noetherian local ring R. Show that

$$\beta_i(I) = \beta_{i+1}(R/I)$$

for all $i \geqslant 0$. In particular, note that I has finite projective dimension if and only if R/I has finite projective dimension.

Problem 3. Check that if M is a free module over a domain, then $\operatorname{rank} M$ is the free rank of M.

Problem 4. Let R be a domain. Show that if M is a finitely generated module with finite projective dimension, then

$$\sum_{i=0}^{\operatorname{pdim}(M)} (-1)^i \beta_i(M) = \operatorname{rank}(M).$$

Problem 5. Show that if I is a proper nonzero ideal of finite projective dimension in a noetherian local domain R, then

$$\sum_{i=0}^{\operatorname{pdim}(R/I)} (-1)^i \beta_i(R/I) = 0.$$

Problem 6. Let $(Q, \mathfrak{m})$ be a regular local ring, $R = Q/I$ with $I \subseteq \mathfrak{m}$ a nonzero ideal in R, and let M be a finitely generated R-module. Show that for any finite free resolution F for M over Q,

$$\sum_{i \geqslant 0} \operatorname{rank} F_{2i} = \sum_{i \geqslant 0} \operatorname{rank} F_{2i+1}.$$

Problem 7. Show that

$$\beta_i(M) = \operatorname{rank} \Omega_i(M) + \operatorname{rank} \Omega_{i+1}(M).$$

Problem 8. Let $Q = k[\![x, y, z, w]\!]$, $I = (xy, yz, zw)$, and $M = Q/I$.

a) Find $\operatorname{pdim}(M)$ without writing the minimal free resolution for M.

b) Find the betti numbers of M without writing the minimal free resolution for M.

c) Find the minimal free resolution for M.

d) Check your work with Macaulay2.

Problem 9. Let M be a finitely generated R-module. Assume that either $(R, \mathfrak{m}, k)$ is a noetherian local ring or that R is a standard graded finitely generated algebra over a field $k = R_0$, in which case M is graded.

a) Show that

$$\beta_i(M) = \dim_k \operatorname{Tor}_i^R(M, k) = \dim_k \operatorname{Ext}_R^i(M, k).$$

b) In the graded case, show that

$$\beta_{i,j}(M) = \dim_k \operatorname{Tor}_i^R(M, k)_j = \dim_k \operatorname{Ext}_R^i(M, k)_{-j}.$$

Problem 10. Let R be a noetherian local ring and let M and N be finitely generated R-modules. Show that for all $i \geqslant 1$,

$$\mathrm{Tor}_{i+1}^R(M,N) \cong \mathrm{Tor}_i^R(\Omega_1 M, N).$$

Problem 11. Let R be a regular local ring. Show that for all prime ideals P, the localization R_P is a regular local ring.

Problem 12. Show that $\beta_2(R/I)$ can be arbitrarily large for 3-generated ideals. More precisely, show that for all $N \geqslant 1$ there exists d and an ideal $I = (f,g,h)$ in $R = k[x_1, \ldots, x_d]$ such that $\beta_2(R/I) \geqslant N$.

Problem 13. Let $M \neq 0$ be a finitely generated module over a noetherian local ring, and let $p = \mathrm{pdim}(M) < \infty$. Show that

$$\beta_i(M) \geqslant \begin{cases} 2i+1 & \text{if } i < p-1 \\ p & \text{if } i = p-1 \\ 1 & \text{if } i = p. \end{cases}$$

Problem 14. Let $I \neq R$ be a radical ideal in a regular ring R, and set

$$c := \max\{\mathrm{height}\, P \mid P \in \mathrm{Min}(I)\}.$$

Show that for all i,

$$\beta_i(R/I) \geqslant \binom{c}{i}.$$

Problem 15. Let R be a noetherian local domain and consider an R-module homomorphism $g\colon R^a \longrightarrow R^b$. Show that if g is injective, then $a \leqslant b$.

Problem 16. Let k be a field and consider an exact sequence of k-vector spaces $A \longrightarrow B \longrightarrow C$. Show that

$$\dim_k B \leqslant \dim_k A + \dim_k C.$$

Problem 17. Let Q be a regular local ring and $0 \neq f \in \mathfrak{m}$. Show that $Q/(f)$ is a regular ring if and only if $f \notin \mathfrak{m}^2$.

Problem 18. Let I be a nonzero proper ideal in a noetherian domain R and let $f_1, \ldots, f_c$ be a maximal regular sequence inside I. Consider the short exact sequence

$$0 \longrightarrow N \longrightarrow R/(f_1, \ldots, f_c) \xrightarrow{\pi} R/I \longrightarrow 0.$$

where π is the canonical quotient map.

a) Show that $\mathrm{Ext}_R^{c-1}(N, R) = 0$.
b) Show that the induced map

$$\pi^* = \mathrm{Ext}_R^c(\pi, R)\colon \mathrm{Ext}_R^c(R/I, R) \longrightarrow \mathrm{Ext}_R^c(R/(f_1, \ldots, f_c), R)$$

is nonzero.

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Problem Set 2

Throughout, all rings all commutative and noetherian, and all modules are finitely generated.

Problem 19. Let $Q = k[\![x,y]\!]$, $I = (x^2, xy)$, and $R = Q/I$.

a) Write the first 3 steps to construct a minimal model for R over Q.

b) Write the first 3 steps to construct an acyclic closure for k over R.

Problem 20. Let $(R, \mathfrak{m}, k)$ be any noetherian local ring of dimension d. Show that

$$\beta_i(k) \geqslant \binom{d}{i}.$$

Problem 21. Let R be a noetherian local ring and P a prime ideal in R. Show that if R is a complete intersection, then so is R_P.

Problem 22. Show that if R is a complete intersection of codimension c, then every finitely generated R-module has complexity at most c.

Problem 23. Let Q be a regular local ring and let $R = Q/I$ with I minimally generated by $\underline{f} = f_1, \ldots, f_n$. Let F be a free resolution of R over Q that has a structure of a DG algebra. Let $e_1, \ldots, e_n$ be a basis for F_1 with $\partial(e_i) = f_i$. Show that we get a system of higher homotopies $\{\sigma_\omega\}$ for $\underline{f}$ on F by setting

$$\sigma_{\mathbf{e}_i}(-) = e_i \cdot - \qquad \text{and} \qquad \sigma_\omega(u) = 0 \text{ for all } |\omega| \geqslant 2.$$

Problem 24. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let F be a complex of finitely generated free R-modules, not necessarily bounded on either side.

a) Show that if f is a regular element on R, then F is exact if and only if $F \otimes_R R/(f)$ is exact.

b) Show that if $\underline{f}$ is a regular sequence, then F is exact if and only if $F \otimes_R R/(\underline{f})$ is exact.

c) Show that if R is regular, then F is exact if and only if $F \otimes_R R/\mathfrak{m}$.

Problem 25. Show that if R is a complete intersection of codimension c, then every finitely generated R-module has complexity at most c.

Problem 26. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let F be a free resolution for the finitely generated R-module M, not necessarily finite. Let $\underline{f} = f_1, \ldots, f_n \in \operatorname{ann}_R(M)$. Show that there exists a system of higher homotopies for $\underline{f}$ on F.

Problem 27. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let F be a free resolution for the finitely generated R-module M, not necessarily finite. Let $\{\sigma_\omega\}$ is a system of higher homotopies for $\underline{f} = f_1, \ldots, f_n$ on F. Show that for all $a_1, \ldots, a_n \in R$ not all zero, the maps

$$\sigma_i := \sum_{|\omega|=i} a_1^{\omega_1} \cdots a_n^{\omega_n} \sigma_\omega$$

form a system of higher homotopies for $a_1 f_1 + \cdots + a_n f_n$ on F.