Homological methods in commutative algebra

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Acknowledgements

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The first of these lectures is heavily based on my survey paper with Adam Boocher on *Lower bounds on Betti numbers* [BG21]. These are in no way comprehensive, but more about these topics can be found in the references — especially in Avramov's excellent *Infinite free resolutions* lecture notes [Avr10].

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Setup

Throughout, all rings are commutative noetherian rings with $1 \neq 0$. We will be primarily be concerned with two main settings:

local setting	graded setting
$(R, \mathfrak{m}, k)$ noetherian local ring	$R = k[x_1, \ldots, x_d]/I$ $k[x_1, \ldots, x_d]$ standard graded, k field, I homogeneous
M is a finitely generated R-module $\mathfrak{m}$ the unique maximal ideal	M is a finitely generated <u>graded</u> R-module $\mathfrak{m} = (x_1, \ldots, x_d)$ unique homogeneous maximal ideal

In the graded settings, we will consider only homogeneous elements and graded modules. In both of these settings, we can use NAK and all its consequences.

1 Free resolutions and Betti numbers

Given an R-module M, how do we describe it? We need to know a set of generators and the relations among those generators. Going further and asking for relations among the relations (treating the relations as generators for the module of relations), and relations among the relations among the relations, and so on, we construct a free resolution for M. Free resolutions play a key role in many important constructions, and encode a lot of interesting information about our module. For example, if the module came from some geometric setting, geometric information about the module gets reflected in the free resolution.

Definition 1.1. Let M a module over a ring R. A **free resolution** of M is a complex of free R-modules F_i

$$F = \cdots \longrightarrow \underset{\mathrm{n}}{F_n} \longrightarrow \cdots \longrightarrow \underset{1}{F_1} \longrightarrow \underset{0}{F_0} \longrightarrow 0$$

together with an isomorphism $\mathrm{H}_0(F) \cong M$, such that $\mathrm{H}_i(F) = 0$ for all $i \neq 0$. We will abuse notation and carelessly identify the resolution for M with the augmented resolution, which is the exact sequence

$$\cdots \longrightarrow \underset{\mathrm{n}}{F_n} \longrightarrow \cdots \longrightarrow \underset{1}{F_1} \longrightarrow \underset{0}{F_0} \longrightarrow M \longrightarrow 0.$$

We can think of a free resolution of M as an approximation of M by free modules. Since every module is a quotient of a free module, every module has a free resolution:

Construction 1.2 (Minimal free resolution)**.** Let M be a finitely generated module over R, where R is either local or graded as in our general setup. If M has β_0 many minimal generators, then we can write a surjective R-module homomorphism from R^{β_0} to M, say

$$R^{\beta_0} \xrightarrow{\pi_0} M.$$

If π_0 is an isomorphism, then $M \cong R^{\beta_0}$ is a **free module** of **rank** β_0. Otherwise, π_0 has a nonzero kernel $\ker(\pi_0)$, which must also be a finitely generated module since R is noetherian. If $\ker(\pi_0)$ is minimally generated by β_1 elements, then we repeat this process and construct a surjective R-module map from R^{β_1} to $\ker(\pi_0)$, and compose it with the inclusion of $\ker(\pi_0)$ into R^{β_0}:

$$\begin{tikzcd}
R^{\beta_1} \arrow[rr, dashed] \arrow[dr] & & R^{\beta_0} \arrow[r, two heads, "\pi_0"] & M. \\
& \ker(\pi_0) \arrow[ur, hook] & &
\end{tikzcd}$$

The elements in $\ker(\pi_0)$ are the **relations** on our chosen minimal generators for M: if M is generated by $m_1, \ldots, m_{\beta_0}$, we can take π_0 to be the map sending each canonical basis element e_i in R^{β_0} to m_i, and an element $(r_1, \ldots, r_{\beta_0}) \in \ker(\pi_0)$ corresponds precisely to a **relation** among the m_i, meaning

$$r_1 m_1 + \cdots + r_{\beta_0} m_{\beta_0} = 0.$$

Such relations are called **syzygies**[1] of M and the module $\ker(\pi_0)$ is the first syzygy module of M, which we will denote by $\Omega_1(M)$.

Continuing this process, we can construct a free resolution for M:

$$\cdots \longrightarrow F_n \xrightarrow{\pi_n} \cdots \xrightarrow{\pi_2} F_1 \xrightarrow{\pi_1} F_0 \xrightarrow{\pi_0} M \longrightarrow 0.$$

At each step, we can choose F_i to have the minimal number of generators; in that case, we say that F is a **minimal free resolution** for M.

One can show the following remarkable facts:

- Every free resolution of M has a minimal free resolution of M as a direct summand.
- Any two minimal free resolutions of M are isomorphic complexes, thus we can talk about *the* minimal free resolution of M.
- As a consequence of the previous facts, the minimal free resolution of M must have the shortest length of any resolution for M, and M has a finite resolution if and only if the minimal free resolution of M is finite.
- A free resolution F for M with differential ∂ is a minimal free resolution for M if and only if $\partial(F) \subseteq \mathfrak{m}F$. Thus if we fix bases for all the free modules F_i, the resolution is minimal if and only if all the entries in the matrices representing ∂ have all entries in $\mathfrak{m}$.

Definition 1.3. Consider a minimal free resolution F of M, and consider the notation in Construction 1.2. The i**th syzygy module of** M, denoted $\Omega_i(M)$, is defined to be the image of π_i, or equivalently the kernel of π_{i-1}.

Note that $\Omega_i(M)$ is defined only up to isomorphism.

Definition 1.4. If a module M has a finite (minimal) free resolution, the length of the minimal free resolution for M is the **projective dimension** of M, and we write it $\operatorname{pdim} M$. Below is a finite resolution of length p:

$$0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

If the minimal free resolution of M is infinite, we say that $\operatorname{pdim} M = \infty$.

Remark 1.5. Suppose that at some point when constructing a resolution following the procedure we described in Construction 1.2, we obtain an injective map of free modules. Then its kernel is trivial, so we obtain a finite free resolution.

Definition 1.6 (Betti numbers)**.** Let F be the minimal free resolution of M. The i**th Betti number** of M is

$$\beta_i(M) := \operatorname{rank}(F_i).$$

Remark 1.7. Note that $\beta_0(M) = \mu(M) = \operatorname{rank}_k(M/\mathfrak{m}M)$.

[1]Fun fact: in astronomy, a syzygy is an alignment of three or more celestial objects.

Exercise 1. Show that
$$\beta_i(M) = \dim_k \operatorname{Tor}_i^R(M,k) = \dim_k \operatorname{Ext}_R^i(M,k).$$

Example 1.8. Let $R = k[\![x,y]\!]$ and $M = R/(x^2, xy)$. Let us write a minimal free resolution for M. First, we note that M is cyclic, so we start with

$$R \longrightarrow R/(x^2,xy) \longrightarrow 0.$$

In this case, the relations on the unique generator 1 in degree 0 are $x^2 \cdot 1 = 0$ and $xy \cdot 1 = 0$, so we proceed with

$$R^2 \xrightarrow{\begin{bmatrix} x^2 & xy \end{bmatrix}} R \longrightarrow R/(x^2,xy) \longrightarrow 0.$$

Now we need to find all relations among x^2 and xy, meaning all choices of a, b such that $ax^2 + bxy = 0$. We note that x is a regular element on R, so $ax^2 = -bxy \implies ax = -by$. But x is regular modulo y, so $a \in (y)$ and $b \in (x)$.[2] Thus

$$\underline{y} \cdot x^2 + \underline{-x} \cdot xy = 0$$

is one of the relations we are looking for, and *all* other such relations are multiples of this one. This shows that we can continue our resolution by taking

$$R \xrightarrow{\begin{bmatrix} y \\ -x \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} x^2 & xy \end{bmatrix}} R \longrightarrow R/(x^2,xy) \longrightarrow 0.$$

Now note that R is a domain, so the leftmost map is in fact injective, and we are done. We conclude that

$$0 \longrightarrow \underset{2}{R} \xrightarrow{\begin{bmatrix} y \\ -x \end{bmatrix}} \underset{1}{R^2} \xrightarrow{\begin{bmatrix} x^2 & xy \end{bmatrix}} \underset{0}{R} \longrightarrow R/(x^2,xy) \longrightarrow 0$$

is a free resolution for $R/(x^2, xy)$. We also took as few generators at each step as possible, so this is a minimal free resolution. We can check this more precisely by noting that all the entries in our matrices are nonunits. In particular, we learn that $\operatorname{pdim}(R/(x^2,xy)) = 2$.

Definition 1.9. Let R be a domain with fraction field Q. The **rank** of a finitely generated R-module M is defined as
$$\operatorname{rank} M := \dim_Q(M \otimes_R Q).$$

Exercise 2. Check that if M is a free R-module, then $\operatorname{rank} M$ is the free rank of M.

Exercise 3. Show that if M has finite projective dimension, then

$$\sum_{i=0}^{\operatorname{pdim}(M)} (-1)^i \beta_i(M) = \operatorname{rank}(M).$$

[2] We will recall the definitions for regular elements and regular sequence very soon.

Theorem 1.10 (Hilbert Syzygy Theorem)**.** *Let $R = k[x_1, \ldots, x_d]$ over a field k. Every finitely generated graded R-module has finite projective dimension, and in fact* $\operatorname{pdim}(M) \leqslant d$*.*

In fact, in the local case, the fact that all finitely generated modules have finite projective dimension *characterizes* regular rings. This characterization is the key ingredient to solve the Localization Problem for regular rings.

Theorem 1.11 (Auslander–Buchsbaum [AB57], Serre [Ser56])**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. The following are equivalent:*

1. *The ring R is regular.*
2. *Every finitely generated R-module has finite projective dimension.*
3. *The residue field k has finite projective dimension.*

Exercise 4 (The Localization Problem for Regular Rings)**.** Let R be a regular local ring. Show that for all prime ideals P, the localization R_P is a regular local ring.

We will discuss singular rings and infinite resolutions later on. For now, let us stick to the case of regular local rings or polynomial rings over a field.

One of our main motivating problems will be to try to understand the shape of minimal free resolutions. As a starting point, let us classify all resolutions of modules with a small number of generators. We begin with cyclic modules, meaning modules of the form $M = R/I$, with I an ideal in R, and take I itself to have a small numbers of generators.

Example 1.12 (Ideals with 1 generator)**.** Let R be a regular local ring of dimension d or a polynomial ring $R = k[x_1, \ldots, x_d]$ over a field k. Since R is a domain, any $0 \neq f \in R$ is a regular element, thus

$$0 \longrightarrow R \xrightarrow{f} R \longrightarrow R/(f) \longrightarrow 0$$

is a minimal free resolution for $R/(f)$.

Note, however, that our assumptions matter:

Example 1.13. Let $R = k[x]/(x^3)$. The minimal free resolution for $k \cong R/(x)$ is

$$\cdots \longrightarrow R \xrightarrow{x} R \xrightarrow{x^2} R \xrightarrow{x} R \longrightarrow R/(x) \longrightarrow 0.$$

Example 1.14 (Ideals with 2 generators)**.** Let R be a regular local ring of dimension d or a polynomial ring $R = k[x_1, \ldots, x_d]$ over a field k. If $I = (f, g) \subseteq (x_1, \ldots, x_d)$ and $c = \gcd(f, g)$, then the minimal free resolution of R/I has length two:

$$0 \longrightarrow R \xrightarrow{\begin{bmatrix} g/c \\ -f/c \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} f & g \end{bmatrix}} R \longrightarrow R/I \longrightarrow 0.$$

When $\gcd(f, g) = 1$, the resolution of $R/(f, g)$ in Example 1.14 is the Koszul complex. The Koszul complex is arguably the most important complex in commutative algebra (and beyond). It appears everywhere, and it is a very powerful yet elementary tool any homological algebraist needs in their toolbox. Every sequence of elements $x_1, \ldots, x_n$ in any ring R gives rise to a Koszul complex.

Construction 1.15 (The Koszul complex)**.** The **Koszul complex** on one element $x \in R$ is the complex

$$\operatorname{kos}(x) := \; 0 \longrightarrow \underset{1}{R} \xrightarrow{\;x\;} \underset{0}{R} \longrightarrow 0.$$

More generally, given $x_1, \ldots, x_n \in R$, the **Koszul complex** with respect to $x_1, \ldots, x_n$ is the complex $\operatorname{kos}(x_1, \ldots, x_n)$ defined inductively as

$$\operatorname{kos}(x_1, \ldots, x_n) := \operatorname{kos}(x_1, \ldots, x_{n-1}) \otimes_R \operatorname{kos}(x_n).$$

You will find different sign conventions for the Koszul complex in the literature, but at the end of the day they all lead to isomorphic complexes.

Example 1.16. The Koszul complex on $f, g \in R$ is given by

$$K_\bullet(f,g) \quad = \text{Totalization of} \quad \begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & R \arrow[r, "-g"] \arrow[d, "f"'] & R \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & R \arrow[r, "g"'] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & \end{tikzcd} \quad = \quad 0 \longrightarrow R \xrightarrow{\binom{-g}{f}} R^2 \xrightarrow{(f \quad g)} R \longrightarrow 0$$

The Koszul complex has more structure than simply being a complex: it is an example of a differentially graded algebra, or DG algebra for short, meaning it has an algebra structure on it as well. We will discuss these in more detail soon; for now we will briefly describe how to construct the Koszul complex in such a way, but emphasize that this is only the beginning of a beautiful story.

In a rare moment of non-commutativity, we will need to consider exterior algebras.

Definition 1.17. The **exterior algebra** $\bigwedge M$ on an R-module M is obtained by taking the the free R-algebra on M, $R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \cdots$, modulo the relations $x \otimes y = -y \otimes x$ and $x \otimes x = 0$ for all $x, y \in M$. We denote the product on $\bigwedge M$ by $a \wedge b$, and see $\bigwedge M$ as a graded algebra where the homogeneous elements in degree d are those in the image of $M^{\otimes d}$. This is a **skew commutative** algebra, since for any homogeneous elements a and b

$$a \wedge b = (-1)^{\deg(a)\deg(b)} b \wedge a \quad \text{and} \quad a \wedge a = 0 \text{ whenever } a \text{ has odd degree.}$$

. We denote the set of all homogeneous elements of degree n by $\bigwedge^n M$. Note also that this construction is functorial: a map $M \xrightarrow{f} N$ of R-modules induces a map $\bigwedge M \xrightarrow{\wedge f} \bigwedge N$ given by $m_1 \wedge \cdots \wedge m_s \mapsto f(m_1) \wedge \cdots \wedge f(m_s)$.

We will use this construction in the case of free modules. When $M = R^n$ with basis $e_1, \ldots, e_n$, then $\wedge^s M \cong R^{\binom{n}{s}}$ for all $1 \leqslant s \leqslant n$, with basis $e_{i_1} \wedge \cdots \wedge e_{i_s}$ ranging over all $i_1 < i_2 < \cdots < i_s$.

Definition 1.18 (The Koszul complex, again)**.** Let $x_1, \ldots, x_n$ be elements in R. The **Koszul complex** on $x_1, \ldots, x_n$ is the complex

$$\operatorname{kos}(x_1, \ldots, x_n) := \; 0 \longrightarrow \bigwedge^n R^n \longrightarrow \bigwedge^{n-1} R^n \longrightarrow \cdots \longrightarrow \bigwedge^1 R^n \longrightarrow R \longrightarrow 0$$

with differential given by

$$d(e_{i_1} \wedge \cdots \wedge e_{i_s}) = \sum_{1 \leqslant p \leqslant s} (-1)^{p-1} x_{i_p} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_s}.$$

More generally, given an R-module M, the Koszul complex on M with respect to $x_1, \ldots, x_n$ is $\operatorname{kos}(x_1, \ldots, x_n; M) := \operatorname{kos}(x_1, \ldots, x_n) \otimes_R M$.

Exercise 5. Show that d as defined above is indeed a differential, meaning $d^2 = 0$.

Exercise 6. Write the Koszul complex on 3 elements f_1, f_2, f_3.

The Koszul complex on $\underline{f} = f_1, \ldots, f_n$ detects whether $\underline{f}$ is a regular sequence.

Definition 1.19 (Regular sequence)**.** An nonunit $f \in R$ is **regular** on an R-module M if $fm \neq 0$ for all nonzero $m \in M$. We say that $\underline{f} = f_1, \ldots, f_n$ is a **regular sequence** on M if

- $(f_1, \ldots, f_n)M \neq M$.
- For each i, f_i is a regular element on $M/(f_1, \ldots, f_{i-1})M$.

We say $\underline{f} = f_1, \ldots, f_n \in \mathfrak{m}$ is a regular sequence if it is a regular sequence on R.

Theorem 1.20. *A noetherian local ring $(R, \mathfrak{m}, k)$ is regular if and only if $\mathfrak{m}$ is generated by a regular sequence.*

Theorem 1.21. *Let R be a local or graded ring, and let $f_1, \ldots, f_n \in R$ be (homogeneous) nonunits. The following are equivalent:*

1. *The elements $f_1, \ldots, f_n$ form a regular sequence.*
2. *The Koszul complex is a resolution of $R/(f_1, \ldots, f_n)$.*
3. *The first koszul homology vanishes:* $\mathrm{H}_1(\operatorname{kos}(f_1, \ldots, f_n)) = 0$.

As a consequence, we see that in our setting, $\underline{f} = f_1, \ldots, f_n$ is a regular sequence if and only any shuffling of the elements is also a regular sequence.

Corollary 1.22. *Let $f_1, \ldots, f_c \in R$ be a regular sequence, and let $I = (f_1, \ldots, f_c)$. Then*

$$\beta_i(R/I) = \binom{c}{i}.$$

Definition 1.23. The **grade** of an ideal I in R, written $\operatorname{grade} I$, is the largest length of a regular sequence inside I. The **depth** of an R-module M, written $\operatorname{depth} M$, is the largest length of a sequence $f_1, \ldots, f_n \in \mathfrak{m}$ of elements that are regular on M.

The following well-known formula is quite useful:

Theorem 1.24 (Auslander–Buchsbaum Formula)**.** *Let R be a local or graded ring (as in our initial setting) and let M be a finitely generated (graded) R-module of finite projective dimension. Then*

$$\operatorname{pdim} M + \operatorname{depth} M = \operatorname{depth} R.$$

Example 1.25. Let $R = k[x,y,z]$ and $M = R/(xy, xz, yz)$. The minimal free resolution for M is

$$0 \longrightarrow R^{2} \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & -x \end{pmatrix}} R^{3} \xrightarrow{\begin{pmatrix} xy & xz & yz \end{pmatrix}} R \longrightarrow M \longrightarrow 0.$$

This is not a Koszul complex, and neither are these the Betti numbers of a Koszul complex; instead, the Betti numbers of M are

$$\beta_0(M) = 1 \quad \beta_1(M) = 3 \quad \beta_2(M) = 2.$$

Of course this is because xy, xz, yz is not a regular sequence.

This is a special case of the Hilbert–Burch Theorem [Bur68], which tells us about the shape of the minimal free resolution of cyclic modules of projective dimension 2.

The ideal in Example 1.25 is homogeneous, and thus we can in fact rethink our resolution in a way that keeps track of the grading, and talk about *graded* Betti numbers of M.

Definition 1.26. Let R be a standard graded k-algebra with $R_0 = k$ and homogeneous maximal ideal $\mathfrak{m} = R_+$. Let M be a graded R-module with minimal graded free resolution F. The (i,j)th Betti number of M, $\beta_{ij}(M)$, counts the number of generators of F_i in degree j. We often collect the Betti numbers of a module in its **Betti table**:

$\beta(M)$	0	1	2	$\cdots$
0	$\beta_{00}(M)$	$\beta_{01}(M)$	$\beta_{02}(M)$	
1	$\beta_{11}(M)$	$\beta_{12}(M)$	$\beta_{13}(M)$	
2	$\beta_{22}(M)$	$\beta_{23}(M)$		
$\vdots$			$\ddots$	

By convention, the entry corresponding to (i,j) in the Betti table of M contains $\beta_{i,i+j}(M)$, and *not* $\beta_{ij}(M)$. This is how Macaulay2 displays Betti tables.

Example 1.27. From the minimal resolution in Example 1.25, we can read the graded Betti numbers of M:

- $\beta_0(M) = 1$, since M is cyclic, and the unique generator lives in degree 0, so $\beta_{0,0}(M) = 1$.
- $\beta_1(M) = 3$, and these three quadratic generators live in degree 2, so $\beta_{12} = 3$.
- $\beta_2(M) = 2$, representing linear syzygies on quadrics, living in degree 1+2=3, so $\beta_{23} = 2$.

To write a graded free resolution for M, we choose all maps to have degree 0, so that the graded free modules in each degree are sums of copies of shifts of R. We write $R(-d)$ for the R-module R but with a new grading, where $(R(-d))_i := R_{i-d}$. Here is the graded free resolution of M:

$$0 \longrightarrow R(-3)^2 \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & -x \end{pmatrix}} R(-2)^3 \xrightarrow{\begin{pmatrix} xy & xz & yz \end{pmatrix}} R \longrightarrow M \longrightarrow 0.$$

Notice that the graded shifts in lower homological degrees affect all the higher homological degrees as well. For example, when we write the map in degree 2, we only need to shift the degree of each generator by 1, but since our map now lands on $R(-2)^3$, we have to bump up degrees from 2 to 3, and write $R(-3)^2$. The graded Betti number $\beta_{ij}(M)$ of M counts the number of copies of $R(-j)$ in homological degree i in our resolution. So again we have

$$\beta_{00} = 1, \beta_{12} = 3, \text{and } \beta_{23} = 2.$$

We can collect the graded Betti numbers of M in its Betti table:

$\beta(M)$	0	1	2
0	1	–	–
1	–	3	2

.

Example 1.28. Let k be a field, $R = k[x, y]$, and consider the ideal

$$I = (x^2, xy, y^3)$$

which has two generators of degree 2 and one of degree 3, so there are graded Betti numbers β_{12} and β_{13}. The minimal free resolution for R/I is

$$0 \longrightarrow \begin{matrix} R(-3)^1 \\ \bigoplus \\ R(-4)^1 \end{matrix} \xrightarrow{\begin{pmatrix} y & 0 \\ -x & y^2 \\ 0 & -x \end{pmatrix}} \begin{matrix} R(-2)^2 \\ \bigoplus \\ R(-3)^1 \end{matrix} \xrightarrow{\begin{pmatrix} x^2 & xy & y^3 \end{pmatrix}} R \longrightarrow R/I.$$

Thus

$$\begin{array}{ll} \beta_{23}(R/I) = 1 & \qquad \beta_{12}(R/I) = 2 \\ \beta_{24}(R/I) = 1 & \qquad \beta_{13}(R/I) = 1 \end{array}$$

and the Betti table of R/I is

$\beta(M)$	0	1	2
0	1	–	–
1	–	2	1
2	–	1	1

.

When R is a graded ring and M and N are graded R-modules, we can compute $\operatorname{Ext}^i_R(M,N)$ using a graded free resolution of M, and thus the Ext-modules inherit an R-graded structure.

Exercise 7. Let R be a standard graded finitely generated algebra over a field $k = R_0$ and let M be a graded R-module. Show that

$$\beta_{i,j}(M) = \dim_k \left(\operatorname{Tor}_i^R(M,k)_j\right) = \dim_k \left(\operatorname{Ext}^i_R(M,k)_{-j}\right).$$

In fact, even if all we know is the Betti numbers of M, there is lots of information we can extract about M. For more about the beautiful theory of free resolutions and syzygies, see [Eis05]. For a detailed treatment of graded free resolutions, see [Pee11].

But back to our attempt at studying the resolutions of ideals with a small number of generators. Unfortunately, even over a polynomial ring over a field, these can be arbitrarily complicated. Even the resolutions of 3-generated ideals can be as long as possible:

Theorem 1.29 (Burch, 1968 [Bur68])**.** *For every $d \geqslant 2$, there exists a three-generated ideal I in $R = k[x_1,\ldots,x_d]$ such that $\operatorname{pdim}(R/I) = d$.*

In fact, every minimal free resolution is the tail of the minimal resolution of a 3-generated ideal:

Theorem 1.30 (Bruns, 1976 [Bru76])**.** *Let $R = k[x_1,\ldots,x_n]$ and*

$$0 \longrightarrow F_n \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be a minimal free resolution of a finitely generated graded R-module M. Then there exists a 3-generated ideal I in R with minimal free resolution

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_3 \longrightarrow F_2' \longrightarrow R^3 \longrightarrow R \longrightarrow R/I \longrightarrow 0\,.$$

Exercise 8. Show that $\beta_2(R/I)$ can be arbitrarily large for 3-generated ideals. More precisely, show that for all $N \geqslant 1$ there exists d and an ideal $I = (f,g,h)$ in $R = k[x_1,\ldots,x_d]$ such that $\beta_2(R/I) \geqslant N$.

These results indicate that the question of how large the Betti numbers of an ideal can be has a pretty devastating answer: as large as you want them to be. But the question of how *small* Betti numbers can be is much more delicate.

Theorem 1.31 (Syzygy Theorem, Evans–Griffith, 1981 [EG81])**.** *Let M be a finitely generated module of finite projective dimension over a noetherian local ring containing a field. If $\Omega_i(M)$ is not free, then*

$$\operatorname{rank}\left(\Omega_i(M)\right) \geqslant i.$$

Exercise 9. Let M be a finitely generated module over a noetherian local ring. Show that

$$\beta_i(M) = \operatorname{rank}\left(\Omega_i(M)\right) + \operatorname{rank}\left(\Omega_{i+1}(M)\right).$$

Exercise 10. Let $M \neq 0$ be a finitely generated module over a noetherian local ring, and let $p = \operatorname{pdim}(M) < \infty$. Show that

$$\beta_i(M) \geqslant \begin{cases} 2i+1 & \text{if } i < p-1 \\ p & \text{if } i = p-1 \\ 1 & \text{if } i = p. \end{cases}$$

However, $\beta_i(M)$ are conjectured to be substantially bigger. The following is a conjecture of Buchsbaum and Eisenbud [BE77] from the late 1970s, asked independently by Horrocks in a collection of problems compiled by Hartshorne [Har79, Problem 24]. The conjecture predicts that the Koszul complex is the smallest free resolution possible. More precisely, the conjecture says that given an ideal I, its resolution should be compared to the Koszul complex on a maximal regular sequence inside I. Since our ring is Cohen-Macaulay, the length of such a sequence is the same as the height of our ideal I. More generally, we want to compare to a regular sequence on $\operatorname{codim}(M)$ many elements.

Definition 1.32. Let M be a finitely generated module over a noetherian ring. The **codimension** of M is

$$\operatorname{codim}(M) := \dim(R) - \dim(R/\operatorname{ann}(M)).$$

Remark 1.33. In our main setting, note that

$$\dim(R/\operatorname{ann}(M)) = \dim R - \operatorname{height}\operatorname{ann}(M)$$

so

$$\operatorname{codim}(M) = \operatorname{height}\operatorname{ann}(M).$$

Conjecture 1.34 (BEH Conjecture)**.** *Let R be either a noetherian local ring or a standard graded k-algebra over a field $k = R_0$. Let M be a nonzero finitely generated (graded) R-module of finite projective dimension and codimension c. Then for all i,*

$$\beta_i(M) \geqslant \binom{c}{i}.$$

Remark 1.35. Note that $\binom{n}{i} = 0$ when $i < 0$ or $i > n$, so the conjecture is only meaningful for i between 0 and c.

Here is a helpful strategy for thinking about this conjecture.

Remark 1.36. Let M be any finitely generated R-module, and let P be any prime containing $\operatorname{ann}(M)$, so that $M_P \neq 0$. Since localization is flat, localizing a minimal free resolution for M gives us a free resolution for M_P, though not necessarily minimal. Thus the Betti numbers can only get smaller:

$$\beta_i^R(M) \geqslant \beta_i^{R_P}(M_P).$$

We can reduce the BEH Conjecture to modules of finite length.

Definition 1.37. An R-module M has **finite length** if it has a finite filtration of the form

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M.$$

where each quotient M_{i+1}/M_i is simple.

Remark 1.38. Over any noetherian ring, it is well-known that a finitely generated module M has finite length if and only if $\dim(M) = 0$. In our main setting, when R is a noetherian local ring or a quotient of a standard graded polynomial ring by a homogeneous ideal, $\dim(M) = 0$ if and only if the unique (homogeneous) maximal ideal $\mathfrak{m}$ is a minimal prime of M. Note that in this case, we necessarily have $\operatorname{depth}(M) = 0$, and thus by the Auslander–Buchsbaum Formula, the projective dimension of M is as large as possible:

$$\operatorname{pdim}(M) = \operatorname{depth}(R) - \operatorname{depth}(M) = \operatorname{depth}(R).$$

In fact, when R is a regular ring of dimension d, we get $\operatorname{pdim}(M) = d$.

Lemma 1.39. *The local ring version of the BEH Conjecture reduces to finite length modules. More precisely, if all finite length modules of finite projective dimension over a Cohen-Macaulay local ring satisfy Conjecture 1.34, then Conjecture 1.34 holds for all finitely generated modules of finite projective dimension over a noetherian local ring.*

Proof. Suppose that we have shown that all finite length modules over a Cohen-Macaulay local ring satisfy Conjecture 1.34. Let M be an arbitrary finitely generated R-module, not necessarily of finite length. Let c be the codimension of M. By Krull's Height Theorem, there must be a minimal prime P of M of height c. Therefore, in R_P the maximal ideal P_P is a minimal prime of M_P, and thus M_P has finite length over R_P. Moreover, R_P is a regular local ring by Exercise 4. Thus

$$\operatorname{codim}(M_P) = \operatorname{height}\operatorname{ann}(M_P) = \operatorname{height}(P_P) = c.$$

By Remark 1.36, we can compare the Betti numbers of M with the Betti numbers of M_P, which satisfy the conjecture:

$$\beta_i^R(M) \geqslant \beta_i^{R_P}(M_P) \geqslant \binom{c}{i}. \qquad \square$$

While the BEH Conjecture remains open, there is some evidence that it might hold. In fact, sometimes one can increase the value c in the BEH Conjecture.

Exercise 11. Let I be a radical ideal in a regular ring, and set

$$c := \max\{\operatorname{height} P \mid P \in \operatorname{Min}(I)\}.$$

Show that for all i,

$$\beta_i(R/I) \geqslant \binom{c}{i}.$$

Theorem 1.40. *The BEH Conjecture holds for all monomial ideals in a polynomial ring $R = k[x_1, \ldots, x_d]$ over a field k.*

Proof. Given any monomial ideal I, there is a process called *polarization* that allows us to construct a squarefree monomial ideal J from I, which might live in a polynomial ring in a larger number of variables. The polarization J of I has the same height and Betti numbers as I, thus to prove the conjecture holds for all monomial ideals, it suffices to show that it holds for all squarefree monomial ideals. But squarefree ideals are radical, so we are done by Exercise 11. $\square$

Suppose that the BEH Conjecture holds. Then for a module M of codimension c,

$$\sum_{i=0}^{c} \beta_i(M) \geqslant \sum_{i=0}^{c} \binom{c}{i} = 2^c.$$

This is known as the **Total Rank Conjecture**, which was settled in 2018 by Walker [Wal17] in characteristic not 2, and later by Walker and VandeBogert in characteristic 2 [VW25].

Theorem 1.41 (Walker, 2018 [Wal17], VandeBogert–Walker, 2025 [VW25])**.** *If $M \neq 0$ is a finitely generated (graded) module of codimension c over $R = k[x_1, \dots, x_n]$ or a regular local ring R, then*

$$\sum_{i=0}^{c} \beta_i(M) \geqslant 2^c.$$

Moreover, if equality holds then $M = R/I$, where I is an ideal generated by a regular sequence of length c.[3]

One may take this as evidence towards the BEH Conjecture. Nevertheless, it remains an open question. One might even look for counterexamples. For example, as noted by Dugger in [Dug00] it is not known whether there can be an ideal I with height 5 and 6 generators in $R = k[x_1, \dots, x_d]$ such that R/I has the following Betti numbers:

$$0 \longrightarrow R^6 \longrightarrow R^{12} \longrightarrow R^{10} \longrightarrow R^9 \longrightarrow R^6 \longrightarrow R^1 \longrightarrow R/I \longrightarrow 0.$$

One might also wonder if once I is not generated by a regular sequence, perhaps the Betti numbers of I might be even larger than those in a Koszul complex on c generators. For example, Adam Boocher has proposed that one might in general be able to do much better, and obtain

$$\sum_i \beta_i(M) \geqslant 2^c + 2^{c-1}.$$

Boocher proved so in work with collaborators [BW20, BS18] in a number of cases.

For more on the BEH Conjecture and other related open questions, and the state of the art as of a few years ago, see [BG21].

While the general BEH conjecture remains open, it is settled in a number of cases. In the next section, we will discuss the case that inspired Buchsbaum and Eisenbud's original conjecture: if the minimal free resolution of M has more structure – if it has the structure of a DG algebra – then M satisfies the BEH Conjecture.

[3] Slogan: if it walks like the Koszul complex and it quacks like the Koszul complex, then it is the Koszul complex.

2 DG algebra resolutions

Some free resolutions come equipped with additional structure, which is often helpful even if the resolution is not minimal.

Definition 2.1. Let R be a noetherian ring. A DG (differential graded) algebra over R is a complex (A, ∂) of R-modules that has a graded commutative algebra structure which is compatible with the differential, as follows:

1. The underlying graded object
$$\bigoplus_{i \in \mathbb{Z}} A_i$$
is a graded commutative R-algebra. Thus A_0 is a ring. Graded commutativity means that for all homogeneous elements $a \in A_i$ and $b \in A_j$,
$$ab = (-1)^{ij} ba \qquad \text{and} \qquad a^2 = 0 \quad \text{whenever } a \text{ has odd degree.}$$
We write $|a| = i$ to indicate that $a \in A_i$.

2. The differential ∂ satisfies the **Leibniz rule**: for all a and b homogeneous with $|a| = i$,
$$\partial(ab) = \partial(a)b + (-1)^i a\partial(b).$$
Therefore, the multiplication induces a map of complexes $A \otimes A \longrightarrow A$.

Remark 2.2. The condition $a^2 = 0$ for a of odd degree is immediate in characteristic not 2, but in characteristic 2 it does not follow from the rest of the definition.

Definition 2.3. Let A be a DG algebra over R. A **DG module** over A is a complex M of R-modules with the structure of a graded module over A, and such that for all $a \in A_i$ and all $m \in M_j$,
$$\partial(am) = \partial(a)m + (-1)^i a\partial(m).$$

A **DG ideal** of A is a DG submodule of A.

Given a DG algebra A, one can easily show that the cycles $\mathrm{Z}(A)$ form a DG subalgebra of A and the boundaries $\mathrm{B}(A)$ form a DG ideal of $Z(A)$.

Definition 2.4. A **homomorphism of DG algebras** between two DG R-algebras is a map of complexes $\varphi\colon A \to B$ that is also a map of graded R-algebras. A **homomorphism of DG modules** is a homomorphism of graded R-modules that is also a map of complexes.

Example 2.5 (The Koszul complex revisited)**.** The canonical example of a DG algebra is the Koszul complex. Indeed, given any commutative ring R and $\underline{f} = f_1, \ldots, f_n \in R$, the Koszul complex $E = \operatorname{kos}(\underline{f})$ is already a graded commutative R-algebra, with the product

$$(a \cdot e_{i_1} \wedge \cdots \wedge e_{i_s}) \cdot (b \cdot e_{j_1} \wedge \cdots \wedge e_{j_t}) = (ab) \cdot e_{i_1} \wedge \cdots \wedge e_{i_s} \wedge e_{j_1} \wedge \cdots \wedge e_{j_t}.$$

The differential in Definition 1.18 is the unique differential with $\partial(e_i) = f_i$ that satisfies the Leibniz rule.

We will be particularly interested in free resolutions with a DG algebra structure.

Remark 2.6. When $M = R/I$, its minimal free resolution F has $F_0 = R$, so F could support the structure of a DG algebra over R. In general, given such a bounded below complex of free R-modules with $F_0 = R$, the biggest challenge in giving it a DG algebra structure is showing that one can construct a product rule that is associative.

Theorem 2.7 (Buchsbaum–Eisenbud, [BE77])**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local domain and let I be an ideal of R of grade c. If there is a DG algebra structure on the minimal free resolution of R/I, then*

$$\beta_i(R/I) \geqslant \binom{c}{i}.$$

Proof. Let $f_1, \ldots, f_c$ be a maximal regular sequence inside I. Let $E = \operatorname{kos}(f_1, \ldots, f_c)$ and let F be the minimal resolution for R/I, which by assumption has a DG algebra structure. First, we will show that there is an injective homomorphism of DG algebras $\varphi \colon E \longrightarrow F$.

Fix $a_1, \ldots, a_n \in F_1$ such that $\partial(a_i) = f_i$. Consider the DG algebra map $\varphi : E \longrightarrow F$ induced by setting $\varphi_0 = \mathrm{id}_R$ and $\varphi(e_i) = a_i$. In general, given a scalar $b \in R$ and $i_1 < \cdots < i_d$, we must have

$$\varphi(b \cdot e_{i_1} \wedge \cdots \wedge e_{i_d}) = b \cdot \varphi(e_{i_1}) \cdots \varphi(e_{i_d}).$$

We claim that φ is injective. Suppose, by contradiction, that there is some nonzero element in $\ker \varphi$. Thus there must be some homogeneous element z in $\ker \varphi$, say of degree s. Perhaps after reordering $f_1, \ldots, f_c$, we may assume that

$$z = b \cdot e_1 \wedge \cdots \wedge e_s + \sum_{\substack{w_1 \leqslant \cdots \leqslant w_s \\ w_s \geqslant s+1}} c_w \cdot e_{w_1} \wedge \cdots \wedge e_{w_s} \in \ker \varphi$$

for some $b \in R$ and some $c_w \in R$. Note that for all w above,

$$(c_w \cdot e_{w_1} \wedge \cdots \wedge e_{w_s}) \cdot (e_{s+1} \wedge \cdots \wedge e_c) = 0.$$

Since $\ker \varphi$ is a DG ideal of E,

$$b \cdot e_1 \wedge \cdots \wedge e_c = (b \cdot e_1 \wedge \cdots \wedge e_s) \cdot (e_{s+1} \wedge \cdots \wedge e_c) = z \cdot (e_{s+1} \wedge \cdots \wedge e_c) \in \ker \varphi.$$

Note that $F_c \cong R^{\beta_c(R/I)}$ and R is a domain, so if $b \neq 0$, then

$$0 = \varphi(b \cdot e_1 \wedge \cdots \wedge e_c) = b \cdot \varphi(e_1 \wedge \cdots \wedge e_c) \implies \varphi(e_1 \wedge \cdots \wedge e_c) = 0.$$

Since $E_c \cong R$ is generated by $e_1 \wedge \cdots \wedge e_c$, we conclude that $\varphi_c \colon E_c \longrightarrow F_c$ must be the zero map. However, we claim that φ_c is nonzero, giving us a contradiction, which will prove the claim that φ must be injective. To see that $\varphi_c \neq 0$, first note that φ is a lift of the canonical quotient map

$$R/(f_1, \ldots, f_c) \xrightarrow{\pi} R/I$$

to a map of complexes $E \longrightarrow F$, so we can use φ to compute $\operatorname{Ext}^c_R(\pi, R)$ via $\operatorname{Hom}_R(\varphi_c, R)$. In Exercise 12, you will show that $\operatorname{Ext}^c_R(\varphi, R) \neq 0$, which shows that $\operatorname{Hom}_R(\varphi_c, R) \neq 0$, and thus $\varphi_c \neq 0$. This completes the proof that φ is injective.

Now that we have shown that φ is injective, the restrictions $\varphi_i \colon E_i \longrightarrow F_i$ of φ to each degree are injective homomorphisms between free modules. By Exercise 13,

$$\operatorname{rank}(F_i) \geqslant \operatorname{rank}(E_i) = \binom{c}{i}.$$

□

As Avramov notes in [Avr81, Proposition 6.4.1], the proof of Theorem 2.7 only requires the minimal free resolution of M to admit a DG module structure over the Koszul complex E.

Exercise 12. Let I be a nonzero proper ideal in a noetherian domain R and let $f_1, \ldots, f_c$ be a maximal regular sequence inside I. Consider the short exact sequence

$$0 \longrightarrow N \longrightarrow R/(f_1, \ldots, f_c) \xrightarrow{\pi} R/I \longrightarrow 0.$$

where π is the canonical quotient map.

(a) Show that $\operatorname{Ext}_R^{c-1}(N, R) = 0$.

(b) Show that the induced map

$$\pi^* = \operatorname{Ext}_R^c(\pi, R) \colon \operatorname{Ext}_R^c(R/I, R) \longrightarrow \operatorname{Ext}_R^c(R/(f_1, \ldots, f_c), R)$$

is nonzero.

Exercise 13. Let R be a noetherian local domain and consider an R-module homomorphism $g \colon R^a \longrightarrow R^b$. Show that if g is injective, then $a \leqslant b$.

Example 2.8. Over any noetherian local ring R, the minimal resolution of any cyclic module $M = R/I$ of projective dimension 1 admits a natural DG algebra structure: the resolution is of the form

$$0 \longrightarrow F_1 \longrightarrow R \longrightarrow 0$$

and by degree reasons we must have $F_1 \cdot F_1 = 0$. We can then take the products on $F_0 \cdot F_0$, $F_0 \cdot F_1$, and $F_1 \cdot F_0$ to be given by the R-module structure of F_1.

Example 2.9. Suppose that R is any noetherian local ring and consider ideal I such that $M = R/I$ has projective dimension 2. The Hilbert–Burch theorem [Bur68] states that if $\mu(I) = n$, there exists an $n \times (n-1)$ matrix A with entries in R and a regular element $a \in R$ such that $I = aJ$, where J is generated by the $n-1$ minors of A, and the minimal free resolution of R/I is

$$0 \longrightarrow R^{n-1} \xrightarrow{A} R^n \longrightarrow R \longrightarrow 0.$$

Set $F_1 = R^n$ and $F_2 = R^{n-1}$. If there is a DG algebra structure on F, then $F_1 \cdot F_2 = 0$, $F_2 \cdot F_1 = 0$, and $F_2 \cdot F_2 = 0$. Take the products involving F_0 to simply follow the R-module structure of each F_i. Finally, we need to define products of elements of degree 1. More precisely, given a basis $e_1, \ldots, e_n$ of F_1 such that $\partial(e_i) = f_i$, we need to define all products of the form $e_i \cdot e_j$ with $i < j$. (Note that $e_i \cdot e_i = 0$ for all i, and $e_j \cdot e_i = -e_i \cdot e_j$.) To do that, fix a basis $b_1, \ldots, b_{n-1}$ for F_2. Write $A_{i,j}^{\ell}$ for the matrix obtained from A by deleting row ℓ and columns i and j.

Herzog [Her74] showed that there exists a unique DG algebra structure on A, given by

$$e_i \cdot e_j := -a \sum_{\ell=1}^{n-1} (-1)^{i+j+\ell} \det\left(A_{i,j}^{\ell}\right) \cdot b_\ell.$$

One can even go further and show explicitly that any length 3 free resolution of a cyclic module admits a DG algebra structure:

Theorem 2.10 (Buchsbaum–Eisenbud, 1977 [BE77])**.** *Let R be a domain. If* $\operatorname{pdim}(R/I) \leqslant 3$*, then the minimal free resolution of R/I admits a DG algebra structure.*

In their original 1977 paper on the subject, Buchsbaum and Eisenbud [BE77] asked whether the minimal free resolution of every cyclic module over a regular local ring admits a DG algebra structure. But across the ocean in Europe, Avramov [Avr81] already knew that the answer was no, building on an example of Khinich.[4] In fact, Avramov gave obstructions to the existence of a DG algebra structure on the minimal free resolution of a cyclic module.

Example 2.11. Let k be any field and $R = k[x, y, z, w]$. The minimal free resolution of

$$R/(x^2, xy, yz, zw, w^2)$$

does not admit a DG algebra structure.

The cyclic module in Example 2.11 has projective dimension 4, showing one cannot extend Theorem 2.10 to longer resolutions unless we add conditions on R/I. Kustin and Miller proved that if R/I is Gorenstein of projective dimension 4, then the minimal free resolution of R/I admits a DG algebra structure [KM85].

Having a DG algebra resolution buys us a lot more than solving the BEH Conjecture. Moreover, if we are willing to forgo minimality, then every cyclic module over a noetherian local ring has a DG algebra resolution. The following construction is due to Tate [Tat57]:

Construction 2.12 (The Tate resolution)**.** Let Q be any noetherian local ring and let $R = Q/I$. We will construct a DG algebra resolution for R in steps, by successively adding variables in each degree to kill homology in the degree below.

Step 0: Consider the complex with Q in degree 0. The homology of this complex is Q in degree 0, while we would like it to be R.

Step 1: Fix a minimal generating set $f_1, \ldots, f_n$ for I and add variables $x_1, \ldots, x_n$ of degree 1 so that $\partial(x_i) = f_i$. We write

$$Q[x_1, \ldots, x_n \mid \partial(x_i) = f_i]$$

to represent the resulting complex, or $Q[X_1]$ with $X_1 = \{x_1, \ldots, x_n\}$ for short. This gives us

$$\underset{1}{\bigoplus_{i=1}^{n} Q \cdot x_i} \xrightarrow{\partial} \underset{0}{Q}$$

just as we would normally start with when building a resolution for R over Q, but these x_i are elements in a DG algebra, so we need to consider their products as well, which live in higher degrees. We take these to be exterior variables, so that the only relations among them are the ones necessary to satisfy the definition of a DG algebra: we have

$$x_i x_j = -x_j x_i \quad \text{and} \quad x_i^2 = 0.$$

[4]Remember, this was way before the internet!

The differential on any other element of $Q[X_1]$ is now completely determined by linearity and the Leibniz rule. In fact, $Q[X_1]$ is simply the Koszul complex on $f_1, \ldots, f_n$:

$$0 \longrightarrow Q \cdot x_1 \cdots x_n \longrightarrow \cdots \longrightarrow \bigoplus_{i<j} Q \cdot x_i x_j \longrightarrow \bigoplus_{i=1}^{n} Q \cdot x_i \xrightarrow{\partial} Q.$$

So far, we have managed to fix the homology in degree 0 to be R. If $\mathrm{H}_1(Q[X_1]) = 0$, then in fact by Theorem 1.21 the koszul complex must be exact, and we have finished constructing a resolution for R. Otherwise, we proceed to step 2.

Step 2: Fix cycles $z_1, \ldots, z_s \in Q[X_1]$ of degree 1 such that their homology classes $[z_1], \ldots, [z_s]$ minimally generate $\mathrm{H}_1(Q[X_1])$, and add variables $x_{n+1}, \ldots, x_{n+s}$ of degree 2 to kill the homology of degree 1, meaning that we set

$$\partial(x_{n+i}) = z_i.$$

We may take these variables of degree 2 to be of one of two kinds: polynomial variables or divided power variables, with the latter being the choice in Tate's original construction. Let us first describe what happens when we take polynomial variables. In this case, there are no additional relations except for the fact that any two variables of degree 2 commute with each other and with all variables of degree 1. The differential of the resulting complex is completely determined by Q-linearity and the Leibniz rule. Setting $X_2 = \{x_{n_1}, \ldots, x_{n+s}\}$, we have

$$\mathrm{H}_0(Q[X_1, X_2]) = R \quad \text{and} \quad \mathrm{H}_1(Q[X_1, X_2]) = 0.$$

We then repeat this process in every degree:

Step d: Given sets of variables $X_1, \ldots, X_{d-1}$ such that

$$\mathrm{H}_0(Q[X_1, X_2, \ldots, X_{d-1}]) = R \quad \text{and} \quad \mathrm{H}_i(Q[X_1, X_2, \ldots, X_{d-1}]) = 0 \text{ for all } i < d-1,$$

we fix cycles $u_1, \ldots, u_t$ of degree $d-1$ in $Q[X_1, X_2, \ldots, X_{d-1}]$ whose classes in homology generate $\mathrm{H}_{d-1}(Q[X_1, X_2, \ldots, X_{d-1}])$, and add new variables $v_1, \ldots, v_t$ of degree d to kill the homology in degree $d-1$:

$$\partial(v_i) = u_i.$$

We set $X_d = \{v_1, \ldots, v_t\}$ and proceed with $Q[X_1, X_2, \ldots, X_d]$. Our new variables satisfy only the relations they must:

- When d is odd, we take the v_i to be exterior variables.
- When d is even, we take the v_i to be polynomial variables (or divided power variables, which we will describe below; we choose one or the other for *all* even degrees at once).

Finally, we set

$$X := \bigcup_{i \geqslant 1} X_i.$$

The resulting $Q[X]$ is a free resolution of R with a DG algebra structure.

Following the construction, every cyclic module over a noetherian local ring has a Tate resolution.

Remark 2.13. We noted in the construction that when I is generated by a regular sequence, we may stop at step 1, as the Koszul complex is a resolution for R/I. On the other hand, if the minimal generators for I do not form a regular sequence, by Theorem 1.21 the Koszul complex is not exact in degree 1, and thus we must add variables of degree 2.

Remark 2.14 (Divided power variables)**.** The disadvantage of polynomial variables is only visible in prime characteristic. Each time we add a new variable x of even degree, its ripple effect is felt forever, as all the powers x^n are nonzero. This is sometimes an advantage: by the time we get to fixing the homology in some degree $d-1$, we might already have elements of degree d, made out of products of variables of smaller degrees, that turn those cycles into boundaries. But in prime characteristic p, we might have added new cycles as well: if x has even degree, the Leibniz rule and a bit of induction give us

$$\partial(x^p) = \partial(x)x^{p-1} + x\partial(x^{p-1}) = p\partial(x)x^{p-1} = 0.$$

To avoid this, in prime characteristic, rather than adding one variable x in even degree, we add an infinite collection of variables $x = x^{(1)}$ and $x^{(i)}$ for all $i \geqslant 1$, satisfying the following rules:

$$x^{(i)}x^{(j)} = \binom{i+j}{i} x^{(i+j)} \qquad \text{and} \qquad \partial(x^{(i+1)}) = x^{(i)}\partial(x).$$

Note however that over a field of characteristic 0, this recipe coincides with adding polynomial variables, as

$$x^{(i)} = \frac{1}{i!}x^i.$$

One sometimes writes $S\langle x\rangle$ for the DG S-algebra obtained by adjoining the divided power variable x to S, to distinguish it from $S[x]$, obtained by adjoining the polynomial variable x.

Recall that by Cohen's Structure Theorem [Coh46], every complete noetherian local ring is a quotient of a regular ring.

Definition 2.15. Let R be a noetherian local ring. A **minimal Cohen presentation** for R consists of a regular local ring $(Q, \mathfrak{m})$, an ideal $I \subseteq \mathfrak{m}^2$, and an isomorphism $\widehat{R} \cong Q/I$, where $\widehat{R}$ stands for the completion of R with respect to $\mathfrak{m}$.

The minimality condition in our definition of minimal Cohen presentation is the requirement that $I \subseteq \mathfrak{m}^2$.

Exercise 14. Let Q be a regular local ring and $f \in \mathfrak{m}$. Show that $Q/(f)$ is a regular ring if and only if $f \notin \mathfrak{m}^2$.

Definition 2.16. A ring R is a **complete intersection** of codimension c if for a minimal Cohen presentation $\widehat{R} \cong Q/I$ for R, the ideal I is generated by a regular sequence of length c. A **hypersurface** is a complete intersection of codimension 1, meaning there is some nonzero $f \in \mathfrak{m}_Q^2$ such that $\widehat{R} \cong Q/(f)$.

One can show that these definitions are independent of the choice of minimal Cohen presentation.

Definition 2.17. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and fix a minimal Cohen presentation $\widehat{R} \cong Q/I$ for R.

- A **minimal model** $Q[X]$ for R is a DG algebra resolution for Q/I over Q, where we adjoin exterior variables in odd degrees and polynomial variables in even degrees, and take the smallest number possible of variables in each degree.
- An **acyclic closure** $R\langle Y \rangle$ for k is a DG algebra resolution for k over R, where we adjoin exterior variables in odd degrees and divided power variables in even degrees, and take the smallest number possible of variables in each degree.

One can show that as long as we add as few variables as possible in each degree, the number of variables we add in each degree is independent of the choices made.

Remark 2.18. When R is a complete intersection, we have seen in Remark 2.13 that the minimal model of R is just the Koszul complex on a minimal generating set for I, which is a minimal free resolution for I. We also noted in Remark 2.13 that when R is not a complete intersection, we must necessarily add variables of degree 2, and thus the minimal model of R is necessarily an infinite resolution. Since Q is regular, then R has finite projective dimension over Q, and thus a minimal model for R cannot be a minimal free resolution.

Example 2.19. Let $R = Q$ be a regular local ring. The maximal ideal $\mathfrak{m}$ of R is generated by a regular sequence $\underline{f}$, so the acyclic closure of k is simply the Koszul complex on $\underline{f}$.

Exercise 15. Let $Q = k[\![x, y]\!]$, $I = (x^2, xy)$, and $R = Q/I$.

a) Write the first 3 steps to construct a minimal model for R over Q.
b) Write the first 3 steps to construct an acyclic closure for k over R.

Theorem 2.20 (Gulliksen, 1968 [Gul68], Schoeller, [Sch67], 1967)**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. An acyclic closure for k is a minimal free resolution for k.*

Exercise 16. Let $(R, \mathfrak{m}, k)$ be any noetherian local ring of dimension d. Show that

$$\beta_i(k) \geqslant \binom{d}{i}.$$

Just like the Betti numbers count the number of generators in each homological degree, there is a DG analogue that counts the number of algebra generators we add in each degree. These are especially important for the acyclic closure of the residue field, given Theorem 2.20.

Definition 2.21. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and $R\langle Y \rangle$ be an acyclic closure of k. The **deviations** of R count the number of variables in each degree:

$$\varepsilon_i(R) := |Y_i|.$$

Remark 2.22. Given the deviations of a local ring R, we can easily compute the Betti numbers of k. Let us illustrate this by computing the first few. First, we know that $\beta_0(k) = 1$. Moreover, the minimal resolution for k is $R\langle Y\rangle$, which has the form

$$\cdots \longrightarrow \bigoplus_{y\in Y_3} Ry \oplus \bigoplus_{\substack{x\in Y_1\\ y\in Y_2}} Rxy \oplus \bigoplus_{\substack{x,y,z\in Y_1\\ \text{all distinct}}} Rxyz \longrightarrow \bigoplus_{y\in Y_2} Ry \oplus \bigoplus_{\substack{x,y\in Y_1\\ x\neq y}} Rxy \longrightarrow \bigoplus_{y\in Y_1} Ry \longrightarrow R.$$

Thus

$$\beta_1(k) = \varepsilon_1(R) \qquad \beta_2(k) = \varepsilon_2(R) + \binom{\varepsilon_1(R)}{2} \qquad \beta_3(k) = \varepsilon_3(R) + \varepsilon_2(R)\varepsilon_1(R) + \binom{\varepsilon_1(R)}{3}.$$

One could easily follow this strategy to find all Betti numbers of k. One noteworthy thing is that in these formulas for $\beta_i(k)$ in terms of $\varepsilon_j(R)$, all the coefficients on $\varepsilon_j(R)$ are positive.

Theorem 2.23 (Avramov, 1984 [Avr84])**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Fix an acyclic closure $R\langle Y\rangle$ for k and a minimal model $Q[X]$ for R. Then for all $i \geqslant 2$,*

$$\varepsilon_i(R) = |Y_i| = |X_{i-1}|.$$

The deviations of R are closely related with the Poincaré series of k.

Definition 2.24. Let R be a noetherian local ring and M a finitely generated R-module. The Poincaré series of M is the power series with integer coefficients given by

$$P_M^R(t) := \sum_{d=0}^{\infty} \beta_d(M)\, t^d.$$

Remark 2.25. When $M = R/I$ is a cyclic R-module, then $\beta_0(R/I) = 1$, and the Poincaré series of R/I has the form

$$1 + \sum_{i=1}^{\infty} b_i t^i.$$

Any power series of this form can be written uniquely as a (possibly infinite) product of the form

$$1 + \sum_{i=1}^{\infty} b_i t^i = \frac{\prod_{i=1}^{\infty} (1 + t^{2i+1})^{e_{2i+1}}}{\prod_{i=1}^{\infty} (1 - t^{2i})^{e_{2i}}}$$

that converges in the (t)-adic topology of $\mathbb{Z}[\![t]\!]$. This can be shown via a quick induction, going modulo (t^n) for each successive n to find e_n, which we leave as an exercise.

Now we claim that when we write the Poincaré series of the residue field k in this form, say

$$P_k^R(t) = \frac{\prod_{i=1}^{\infty} (1 + t^{2i+1})^{e_{2i+1}}}{\prod_{i=1}^{\infty} (1 - t^{2i})^{e_{2i}}},$$

these exponents e_n are precisely the deviations $\varepsilon_n(R)$ of R.

To see this, let $R\langle Y\rangle$ be an acyclic closure for k. By Theorem 2.20, this is a minimal free resolution, so the differential in
$$k\langle Y\rangle := R\langle Y\rangle \otimes_R k$$
vanishes. Note that
$$k\langle Y\rangle = \bigotimes_{y\in Y} k\langle y\rangle.$$
Fix a particular variable $y \in Y$. If y has odd degree $2i-1$, then $k\langle y\rangle$ has a copy of k in degree 0 and another in degree $2i-1$, and nothing else, so
$$\sum_{n=0}^{\infty} \operatorname{rank}_k(k\langle y\rangle_n)\cdot t^n = 1 + t^{2i-1}.$$
If y has even degree $2i$, then $k\langle y\rangle = k\langle y^{(i)} \mid i \geqslant 1\rangle$ has one copy of k in every degree that is a multiple of $2i$, and
$$\sum_{n=0}^{\infty} \operatorname{rank}_k(k\langle y\rangle_n)\cdot t^n = \sum_{\ell=0}^{\infty} t^{(2i)\ell} = \frac{1}{1-t^{2i}}.$$
To count the rank of $k\langle Y\rangle$ in degree n, we need only to count the number of monomials in the variables of Y of total degree n. Thus
$$P_k^R(t) = \frac{\displaystyle\prod_{i=1}^{\infty}(1+t^{2i+1})^{|Y_{2i+1}|}}{\displaystyle\prod_{i=1}^{\infty}(1-t^{2i})^{|Y_{2i}|}} = \frac{\displaystyle\prod_{i=1}^{\infty}(1+t^{2i+1})^{\varepsilon_{2i+1}(R)}}{\displaystyle\prod_{i=1}^{\infty}(1-t^{2i})^{\varepsilon_{2i}(R)}}.$$

Remark 2.26. Let $(R,\mathfrak{m},k)$ be a noetherian local ring and consider its completion $\widehat{R}$ (at the maximal ideal $\mathfrak{m}$). Since completion is flat, we can take a minimal free resolution of k over R and tensor it with $\widehat{R}$ over R to obtain a free resolution over $\widehat{R}$, which is still minimal. Thus $\beta_i^R(k) = \beta_i^{\widehat{R}}(k)$ for all i, which given the uniqueness of the product decomposition in Remark 2.25 gives us that $\varepsilon_i(R) = \varepsilon_i(\widehat{R})$ for all i.

We can now think about deviations in two ways: via the acyclic closure of k or via the minimal model of R.

Remark 2.27. Let us compute the first few deviations of a noetherian local ring $(R,\mathfrak{m},k)$. By Remark 2.22,
$$\varepsilon_1(R) = \beta_1(k) = \mu(\mathfrak{m}) = \operatorname{embdim}(R).$$
By Theorem 2.23, if $Q[X]$ is a minimal model for R with $\widehat{R} \cong Q/I$, then
$$\varepsilon_2(R) = |X_1| = \mu(I).$$
Since $Q[X_1]$ is the Koszul complex on a minimal generating set $\underline{f}$ for I, the number of variables in X_2 is the minimal number of generators for the first Koszul homology on $\underline{f}$. Since the Koszul homology is independent of the choice of minimal generators for I, we simply write this as $\mathrm{H}_1(I)$. Thus
$$\varepsilon_3(R) = \mu(\mathrm{H}_1(I)).$$

Lemma 2.28. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. The following are equivalent:*

1. *R is regular.*
2. *$\varepsilon_n(R) = 0$ for all $n \geqslant 2$.*
3. *$\varepsilon_2(R) = 0$.*

Proof. If R is regular, the maximal ideal is generated by a regular sequence, and an acyclic closure of k is just the Koszul complex on a minimal generating set, so there are no variables of degree above 1 and $\varepsilon_n(R) = 0$ for all $n \geqslant 2$. This shows $1 \Rightarrow 2$, and $2 \Rightarrow 3$ is obvious.

If $\varepsilon_2(R) = 0$, then the Koszul complex $R\langle Y_1 \rangle$ on a minimal generating set for $\mathfrak{m}$ has $\mathrm{H}_1(R\langle Y_1 \rangle) = 0$, so by Theorem 1.21 it must be exact. Thus $\mathfrak{m}$ is generated by a regular sequence, and R is regular. Alternatively, we can see that $I = 0$ by Remark 2.27, so $\widehat{R} \cong Q$ is a regular ring. □

The following characterization of complete intersections puts together the work of several people, showing various conditions are equivalent to being a complete intersection: Assmus [Ass59] showed the equivalence with (3) and Gulliksen showed the equivalence with conditions (4) [Gul71] and (5) [Gul80]. The last condition, due to Halperin [Hal87], is the most amazing: as long as one deviation vanishes, then R must be a complete intersection. This tells us that as long as R is not a complete intersection, then when constructing a minimal model for R over Q or an acyclic closure for k over R, we must add new variables in *every* degree.

Theorem 2.29 (Assmus, 1959 [Ass59], Gulliksen, 1971 [Gul71] and 1980 [Gul80], Halperin, 1987 [Hal87])**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. The following are equivalent:*

(1) R is a complete intersection.

(2) $\varepsilon_n(R) = 0$ for all $n \geqslant 3$.

(3) $\varepsilon_3(R) = 0$.

(4) $\varepsilon_n(R) = 0$ for all $n \gg 0$.

(5) $\varepsilon_{2n}(R) = 0$ for all $n \gg 0$.

(6) $\varepsilon_n(R) = 0$ for some $n \geqslant 1$.

We will use this characterization to prove that the complete intersection property, like regularity, localizes. The first proof of the Localization Problem for complete intersections is due to Avramov [Avr77] in 1977. We will give a different proof, due to Gulliksen in 1980, that uses the characterization of complete intersections from above to give yet another equivalent definition of complete intersection.

Remark 2.30. When $(R, \mathfrak{m}, k)$ is a complete local ring, the Localization Problem for complete intersections is very easy to prove: it is simply the statement that if I is an ideal generated by a regular sequence (in a regular ring Q), then so is I_P for all primes $P \supseteq I$. The real difficulty is in the case when R is not complete: given a prime ideal P in R and a prime ideal Q of $\widehat{R}$ such that $Q \cap R = P$, as noted above it is easy to show that if $\widehat{R}$ is a quotient of a regular ring by a regular sequence then so is $\widehat{R}_Q$, but we actually need to show that $\widehat{R_P}$ is also a quotient of a regular ring by a regular sequence.

To solve the Localization Problem, we will first prove yet another characterization of complete intersections, using complexity.

Definition 2.31. We say that a finitely generated R-module M has **finite complexity** if there is a polynomial $f \in \mathbb{Z}[t]$ such that

$$\beta_i(M) \leqslant f(i)$$

for all i. If no such polynomial exists, we say M has **infinite complexity**.

We can even give a value to the complexity of a module.

Definition 2.32. Let M be a finitely generated R-module of finite complexity. If M has finite projective dimension, we say M has complexity 0, and write $\operatorname{cx}(M) = 0$. Otherwise, we say M has complexity d, and write $\operatorname{cx}(M) = d$, if $d-1$ is the smallest degree of a polynomial f with $\beta_i(M) \leqslant f(i)$ for all i.

Remark 2.33. Some authors define the complexity of M to be the least integer d such that $\beta_n(M) \leqslant C \cdot n^{d-1}$ for some constant C and all $n \gg 0$. This has the advantage that includes complexity 0 as part of the definition. Also, note that this is equivalent to our definition, as only the largest power of the polynomial really matters, and when $\operatorname{cx}(M) > 0$ we can always change the constant C so that the inequality applies to all (not just large) n.

Example 2.34. Complexity 1 means that M has infinite projective dimension but the Betti numbers are bounded above by a constant. For example, over $R = k[\![x]\!]/(x^2)$, the module k has complexity 1, as $\beta_i(k) = 1$ for all i.

Theorem 2.35 (Gulliksen, 1980 [Gul80])**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and let $x \in \mathfrak{m}$ be a regular element. Let M be any module over $R/(x)$. Then M has finite complexity over R if and only if M has finite complexity over $R/(x)$.*

Proof. Let $\pi : R \longrightarrow R/(x)$ be the canonical projection. There is a well-known change of rings long exact sequence

$$\cdots \longrightarrow \operatorname{Tor}_{i-1}^{R/(x)}(M,k) \longrightarrow \operatorname{Tor}_i^R(M,k) \xrightarrow{\pi_*} \operatorname{Tor}_i^{R/(x)}(M,k) \longrightarrow \operatorname{Tor}_{i-2}^{R/(x)}(M,k) \longrightarrow \cdots$$

The k-vector space dimensions of these Tor-modules are the Betti numbers of M over R and $R/(x)$, by Exercise 1.

Applying Exercise 17 below with $\operatorname{Tor}_i^R(M,k)$ in the middle, we get

$$\beta_i^R(M) \leqslant \beta_i^{R/(x)}(M) + \beta_{i-1}^{R/(x)}(M).$$

If M has finite complexity over $R/(x)$, say with the Betti numbers bounded above by a polynomial f, then we get

$$\beta_i^R(M) \leqslant f(i) + f(i-1),$$

and setting $g(i) := f(i) + f(i-1)$ gives us a polynomial g of degree $\deg(f)$, so M has finite complexity over R.

Now suppose M has finite complexity over R, say with $\beta_i(M) \leqslant f(i)$ for all i, with f a polynomial, and let us we apply Exercise 17 again, but this time with $\operatorname{Tor}_i^{R/(x)}(M,k)$ in the middle:

$$\beta_i^{R/(x)}(M) \leqslant \beta_i^R(M) + \beta_{i-2}^{R/(x)}(M).$$

Repeating this on $i-2$, and so on, we get

$$\beta_i^{R/(x)}(M) \leqslant \beta_i^R(M) + \beta_{i-2}^R(M) + \beta_{i-4}^R(M) + \cdots = \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} \beta_{i-2j}^R(M) \leqslant \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} f(i-2j),$$

which is a polynomial in i. We conclude that M must also have finite complexity over $R/(x)$. $\square$

Exercise 17. Let k be a field and consider an exact sequence of k-vector spaces $A \longrightarrow B \longrightarrow C$. Show that

$$\dim_k B \leqslant \dim_k A + \dim_k C.$$

Notation 2.36. The symbol $\succeq$ used between two power series with integer coefficients

$$\sum_{i=0}^{\infty} a_i \succeq \sum_{i=0}^{\infty} b_i$$

indicates a coefficientwise inequality, meaning that $a_i \geqslant b_i$ for all i.

Theorem 2.37 (Gulliksen, 1980 [Gul80])**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. The following are equivalent:*

(1) R is a complete intersection.

(2) Every finitely generated R-module has finite complexity.

(3) The residue field k has finite complexity.

Proof. First, note that for all finitely generated R-modules M, the Betti numbers of M and $\widehat{M} = M \otimes_R \widehat{R}$ coincide: tensoring any minimal free resolution of M over R with $\widehat{R}$ gives a minimal free resolution of $\widehat{M}$ over $\widehat{R}$. Thus if every $\widehat{R}$-module has finite complexity, then so does every R-module.

Suppose R is a complete intersection, so that $\widehat{R} \cong Q/I$ for some regular ring Q and ideal I generated by a regular sequence $\underline{f} = f_1, \ldots, f_n$. Since Q is regular, every finitely generated $\widehat{R}$-module has finite projective dimension over Q, and thus (finite) complexity zero over Q. By applying Theorem 2.35 n times, we conclude that every finitely generated $\widehat{R}$-module has finite complexity. As noted above, this shows that every R-module has finite complexity. This shows (1) $\Rightarrow$ (2), and (2) $\Rightarrow$ (3) is trivial; we will now show (3) $\Rightarrow$ (1).

Now suppose that k has finite complexity over R. Since $\beta_i^R(k) = \beta_i^{\widehat{R}}(k)$, we may as well assume that R is complete. Consider the Poincaré series of k, and to simplify notation let us write $\varepsilon_n := \varepsilon_n(R)$. By Remark 2.25,

$$P_k^R(t) = \frac{\displaystyle\prod_{i=1}^{\infty} (1+t^{2i+1})^{\varepsilon_{2i+1}}}{\displaystyle\prod_{i=1}^{\infty} (1-t^{2i})^{\varepsilon_{2i}}}.$$

Suppose that there are least $d+1$ distinct such factors (not counting multiplicities) in the denominator; more precisely, assume that there exist distinct $q_1, \ldots, q_{d+1}$ such that

$$\varepsilon_{2q_1} \geqslant 1, \cdots, \varepsilon_{2q_{d+1}} \geqslant 1.$$

For all integers $a, b \geqslant 1$,

$$\frac{1}{1-t^a} = \sum_{i=0}^{\infty} t^{ai} \succeq \sum_{i=0}^{\infty} t^{abi} = \frac{1}{1-t^{ab}}.$$

Moreover, given two power series of the form

$$1 + \sum_{i=1}^{\infty} a_i t^i$$

with nonnegative integer coefficients, their product is always coefficientwise bounded below by 1; this holds even for an infinite product of such power series, as long as the product converges. Thus setting

$$N := \operatorname{lcm}(2q_1, \ldots, 2q_{d+1}).$$

we conclude that

$$P_k^R(t) \succeq \frac{1}{(1-t^{2q_1}) \cdots (1-t^{2q_{d+1}})} \succeq \frac{1}{(1-t^N)^{d+1}} = \sum_{i=0}^{\infty} \binom{i+d}{i} t^{Ni}.$$

Suppose that g is a polynomial such that $\beta_i(M) \leqslant g(i)$ for all i. Then in particular,

$$g(Ni) \geqslant \beta_{Ni}(k) \geqslant \binom{i+d}{i} = \frac{(i+d)\cdots(i+1)}{d!} \geqslant \frac{(Ni)^d}{N^d d!}.$$

Then g must be a polynomial of degree at least d, so $\operatorname{cx}(k) \geqslant d+1$. Since $\operatorname{cx}(k) < \infty$, we conclude that there can only be finitely many distinct factors in the denominator of $P_k^R(t)$, meaning that $\varepsilon_{2n}(R) = 0$ for all $n \gg 0$. By Theorem 2.29, R must be a complete intersection. $\square$

We can now use this characterization of complete intersections to give a proof of Avramov's result that complete intersections localize; this proof is due to Gulliksen [Gul80].

Exercise 18 (Localization Problem for complete intersections)**.** Let R be a noetherian local ring and P a prime ideal in R. Show that if R is a complete intersection, then so is R_P.

Exercise 19. Show that if R is a complete intersection of codimension c, then every finitely generated R-module has complexity at most c.

Whenever R is not a complete intersection, the resolutions of modules over R can be wild; in particular, the resolution of k grows wildly. In the next section, we will study the resolutions of modules over complete intersections a bit more closely.

3 Resolutions over complete intersections

While resolutions of modules over complete intersections can be infinite, we can construct them using a finite set of data.

Definition 3.1 (Systems of higher homotopies)**.** Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let F be a free resolution for the finitely generated R-module M, not necessarily finite. Let $f \in \operatorname{ann}_R(M)$. A **system of higher homotopies** for f on F is a collection of R-linear maps σ_i of degree $2i-1$

$$\sigma_i \colon F_\bullet \longrightarrow F_{\bullet+2i-1}$$

satisfying the following conditions:

$$\sigma_0 = \partial_F \qquad\qquad \sigma_1\sigma_0 + \sigma_0\sigma_1 = f \cdot \mathrm{id}_F \qquad\qquad \sum_{i=0}^{n} \sigma_i \sigma_{n-i} = 0 \text{ for all } n \geqslant 2.$$

For example, here are depictions of σ_0, σ_1, and σ_2:

$$\begin{tikzcd}
\cdots \arrow[r] & F_3 \arrow[r, "\sigma_0"] & F_2 \arrow[r, "\sigma_0"] \arrow[dl, "\sigma_1"] & F_1 \arrow[r, "\sigma_0"] \arrow[dl, "\sigma_1"] & F_0 \arrow[r] \arrow[dl, "\sigma_1"] & 0 \\
\cdots \arrow[r] & F_3 \arrow[r, "\sigma_0"'] & F_2 \arrow[r, "\sigma_0"'] & F_1 \arrow[r, "\sigma_0"'] & F_0 \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdots \arrow[r] & F_4 \arrow[r] & F_3 \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] \arrow[dlll, "\sigma_2"'] & F_0 \arrow[dlll, "\sigma_2"] \\
\cdots \arrow[r] & F_4 \arrow[r] & F_3 \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0
\end{tikzcd}$$

Remark 3.2 (Systems of higher homotopies exist)**.** If $fM = 0$, then the map $f \cdot \mathrm{id}_F$ is a lift of the zero map $M \longrightarrow M$ to F. Since any two lifts of the same map on M to F must be homotopic, we conclude that $f \cdot \mathrm{id}_F$ must be nullhomotopic. The condition

$$\sigma_1\sigma_0 + \sigma_0\sigma_1 = f \cdot \mathrm{id}_F$$

says that σ_1 is a nullhomotopy for $f \cdot \mathrm{id}_F$. In particular, we can always choose such a map σ_1. Moreover,

$$\sum_{i=0}^{n} \sigma_i \sigma_{n-i} = 0 \iff \sigma_n \partial + \partial \sigma_n = -\sum_{i=1}^{n-1} \sigma_i \sigma_{n-i} =: \tau_n$$

says that σ_n is a nullhomotopy for τ_n. Given $\sigma_0, \ldots, \sigma_{n-1}$ satisfying our desired properties with $n \geqslant 2$, note that

$$\partial \sigma_i = \tau_i - \sigma_i \partial,$$

so

$$\partial \sigma_i \sigma_{n-i} = \tau_i \sigma_{n-i} - \sigma_i \partial \sigma_{n-i} = \tau_i \sigma_{n-i} - \sigma_i(\tau_{n-i} - \sigma_{n-i}\partial) = \tau_i \sigma_{n-i} - \sigma_i \tau_{n-i} + \sigma_i \sigma_{n-i} \partial$$

and

$$\partial \tau_n = -\sum_{i=1}^{n-1} \partial \sigma_i \sigma_{n-i} = \sum_{i=1}^{n-1} (\sigma_i \tau_{n-i} - \tau_i \sigma_{n-i} - \sigma_i \sigma_{n-i} \partial) = \sum_{i=1}^{n-1} (\sigma_i \tau_{n-i} - \tau_i \sigma_{n-i}) + \tau_n \partial.$$

Since

$$\sum_{i=1}^{n-1} (\sigma_i \tau_{n-i} - \tau_i \sigma_{n-i}) = -\sum_{i=1}^{n-1} \sum_{j=1}^{n-i-1} \sigma_i \sigma_j \sigma_{n-i-j} + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i-1} \sigma_j \sigma_{n-i-j} \sigma_{n-i} = 0$$

we conclude that $\partial \tau_n = \tau_n \partial$.

Thus τ_n is a cycle in $\operatorname{Hom}_Q(F,F)$, and it has degree $2n-2 \geqslant 2$. On the other hand, the quasi-isomorphism $F \xrightarrow{\cong} M$ induces a quasi-isomorphism $\operatorname{Hom}_Q(F,F) \xrightarrow{\cong} \operatorname{Hom}_Q(F,M)$, so τ_n corresponds to a cycle of degree $2n-2$ in $\operatorname{Hom}_Q(F,M)$. But $F_{<0} = 0$, so $\operatorname{Hom}_Q(F,M)$ is concentrated in negative degrees, and thus τ_n must be a boundary in $\operatorname{Hom}_Q(F,F)$. Thus there is a nullhomotopy σ_n of degree $2n-1$ for τ_n.

Note, however, that there are many choices along the way, so a system of higher homotopies exists but is not unique.

Example 3.3. Let $Q = k[\![x,y]\!]$ and $M = Q/(x^2, xy)$, and note that x^2 annihilates M. Let us construct a system of higher homotopies for x^2 on the minimal free resolution F for M over Q we wrote in Example 1.8. First, we take σ_0 to be the differential on F. Next, we construct σ_1, which has degree 1:

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r, "\begin{bmatrix} y \\ -x \end{bmatrix}"] \arrow[d, dashed, "\cdot x^2"'] & Q^2 \arrow[r, "\begin{bmatrix} x^2 & xy \end{bmatrix}"] \arrow[d, dashed, "\cdot x^2"'] \arrow[dl, "\begin{bmatrix} a & b \end{bmatrix}"'] & Q \arrow[r] \arrow[d, dashed, "\cdot x^2"] \arrow[dl, "\begin{bmatrix} c \\ d \end{bmatrix}"'] & 0 \\
0 \arrow[r] & Q \arrow[r, "\begin{bmatrix} y \\ -x \end{bmatrix}"'] & Q^2 \arrow[r, "\begin{bmatrix} x^2 & xy \end{bmatrix}"'] & Q \arrow[r] & 0
\end{tikzcd}$$

We must have $cx^2 + dxy = x^2$, so we can take $c = 1$ and $d = 0$. Moreover, looking at the images of $(1,0)$ and $(0,1)$ in homological degree 1, we need

$$ay + x^2 = x^2 \qquad \text{and} \qquad -bx = x^2,$$

so take $a = 0$ and $b = -x$, giving us the following σ_1:

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r, "\begin{bmatrix} y \\ -x \end{bmatrix}"] & Q^2 \arrow[r, "\begin{bmatrix} x^2 & xy \end{bmatrix}"] \arrow[dl, "\begin{bmatrix} 0 & -x \end{bmatrix}"'] & Q \arrow[r] \arrow[dl, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"'] & 0 \\
0 \arrow[r] & Q \arrow[r, "\begin{bmatrix} y \\ -x \end{bmatrix}"'] & Q^2 \arrow[r, "\begin{bmatrix} x^2 & xy \end{bmatrix}"'] & Q \arrow[r] & 0
\end{tikzcd}$$

Now σ_2 would be of degree 3, but F only has length 2, so all the higher homotopies vanish.

Exercise 20. Find a system of higher homotopies for multiplication by xy on the minimal free resolution F in Example 3.3.

We can use systems of higher homotopies to compute free resolutions for modules over complete intersections.

Theorem 3.4 (Shamash, 1969 [Sha69])**.** *Let $(Q, \mathfrak{m}, k)$ be a noetherian local ring, $f \in \mathfrak{m}$ be a regular element, and $R = Q/(f)$. Let M be an R-module, and F be a free resolution for M over Q. Let $\{\sigma_i\}$ be a system of higher homotopies for f on F. Fix symbols $x^{(i)}$ for each integer i, and set $x^{(0)} = 1$ and $x^{(i)} = 0$ whenever $i < 0$. The following is a free resolution for M over R:*

$$\cdots \longrightarrow \underbrace{\bigoplus_{i \geqslant 0} Rx^{(i)} \otimes_Q F_{n-2i}}_{n} \xrightarrow{\partial} \underbrace{\bigoplus_{i \geqslant 0} Rx^{(i)} \otimes_Q F_{n-1-2i}}_{n-1}$$

with differential

$$\partial\left(x^{(i)} \otimes u\right) = \sum_{j=0}^{i} x^{(i-j)} \otimes \sigma_j(u).$$

This is known as the **Shamash construction**.

Exercise 21. Let $R = k[\![x, y]\!]/(x^2)$. Use the Shamash construction to find a resolution for $M = R/(xy)$ over R.

Remark 3.5. Let Q be a regular local ring of dimension d and take any nonzero element $f \neq 0$. Let M be a finitely generated R-module, where $R = Q/(f)$. Since R is regular, $\operatorname{pdim} M \leqslant d$. Let F be the minimal free resolution of M over Q:

$$0 \longrightarrow F_d \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0.$$

Let us apply the Shamash construction to F. Note that in even degrees, we only use F_i with i even, and in odd degrees with we will only use F_i with i odd. The resolution starts with

$$\cdots \longrightarrow \underbrace{\begin{matrix} Rx^{(0)} \otimes_Q F_3 \\ \oplus \\ Rx^{(1)} \otimes_Q F_1 \end{matrix}}_{3} \longrightarrow \underbrace{\begin{matrix} Rx^{(0)} \otimes_Q F_2 \\ \oplus \\ Rx^{(1)} \otimes_Q F_0 \end{matrix}}_{2} \longrightarrow \underbrace{Rx^{(0)} \otimes_Q F_1}_{1} \xrightarrow{\partial} \underbrace{Rx^{(0)} \otimes_Q F_0}_{0} \longrightarrow 0$$

But in high enough degrees (meaning, above degree d), the resolution starts repeating itself. Let us see this in the case when $d = 2a$ is even:

$$\cdots \longrightarrow \underbrace{\begin{matrix} Rx^{(b+1)} \otimes_Q F_{2a} \\ \oplus \\ Rx^{(b+2)} \otimes_Q F_{2a-2} \\ \oplus \\ \vdots \\ \oplus \\ Rx^{(b+1+a)} \otimes_Q F_0 \end{matrix}}_{2a+2b+2} \longrightarrow \underbrace{\begin{matrix} Rx^{(b)} \otimes_Q F_{2a-1} \\ \oplus \\ Rx^{(b+1)} \otimes_Q F_{2a-3} \\ \oplus \\ \vdots \\ \oplus \\ Rx^{(a+b)} \otimes_Q F_1 \end{matrix}}_{2a+2b+1} \longrightarrow \underbrace{\begin{matrix} Rx^{(b)} \otimes_Q F_{2a} \\ \oplus \\ Rx^{(b+1)} \otimes_Q F_{2a-2} \\ \oplus \\ \vdots \\ \oplus \\ Rx^{(b+a)} \otimes_Q F_0 \end{matrix}}_{2a+2b} \longrightarrow \cdots$$

Note that for any $b \geqslant 1$ and $0 \leqslant i \leqslant a$, and any $u \in F_{2a-1-i}$,

$$\partial(x^{(b+i)} \otimes u) = \sum_{j=0}^{b+i} x^{(b+i-j)} \otimes \sigma_j(u).$$

Similarly, for any $b \geqslant 1$ and $0 \leqslant i \leqslant a$, and any $u \in F_{2a-i}$,

$$\partial(x^{(b+i)} \otimes u) = \sum_{j=0}^{b+i} x^{(b+i-j)} \otimes \sigma_j(u).$$

A very similar calculation shows that when $d = 2a+1$, the resolution becomes 2-periodic above degree d.

Let us call the resolution we obtained via this process G. Now note that this is *not* necessarily a minimal free resolution for M over R, but the minimal free resolution for M does split off as a free summand. Since G is eventually 2-periodic, so is the minimal resolution.

In fact, we can use this idea to determine finite projective dimension over a hypersurface.

Lemma 3.6. *Let Q be a regular local ring and $R = Q/(f)$ with $f \neq 0$, and let M be a finitely generated R-module. Let F be a free resolution of M over Q, and $\{\sigma_i\}$ be a system of higher homotopies for f on F. Consider the 2-periodic complexes*

$$P := \quad \cdots \longrightarrow \bigoplus_{i \geqslant 0} F_{2i} \otimes_Q k \xrightarrow{\partial} \bigoplus_{i \geqslant 0} F_{2i+1} \otimes_Q k \xrightarrow{\partial} \bigoplus_{i \geqslant 0} F_{2i} \otimes_Q k \longrightarrow \cdots$$

with

$$\partial(u \otimes 1) = \sum_j \sigma_j(u) \otimes 1$$

and

$$P^* := \quad \cdots \longrightarrow \bigoplus_{i \geqslant 0} \operatorname{Hom}_Q(F_{2i}, k) \xrightarrow{\partial} \bigoplus_{i \geqslant 0} \operatorname{Hom}_Q(F_{2i+1}, k) \xrightarrow{\partial} \bigoplus_{i \geqslant 0} \operatorname{Hom}_Q(F_{2i}, k) \longrightarrow \cdots$$

with

$$\partial = \operatorname{Hom}_Q\left(\sum_j \sigma_j, k\right).$$

Then $\operatorname{pdim}_R(M) < \infty$ if and only if P is exact if and only if P^* is exact.

Proof. Let G be the resolution for M over R given by the Shamash construction we described in Theorem 3.4. By Exercise 1,

$$\beta_i^R(M) = \dim_k \operatorname{Tor}_i^R(M, k),$$

so M has finite projective dimension if and only if $\operatorname{Tor}_i^R(M, k) = 0$ for $i \gg 0$. Thus we need only to consider the tail of the complex $G \otimes_R k$. We described the tail of G in Remark 3.5, and it follows immediately from that description that the tail of $G \otimes_R k$ is the complex P above. Similarly, Exercise 1 also says

$$\beta_i^R(M) = \dim_k \operatorname{Ext}_R^i(M, k),$$

and the tail of $\operatorname{Hom}_Q(G, k)$ is the complex P^* above.

Alternatively, note that P^* is the k-linear dual of P, so P is exact if and only if P^* is exact. $\square$

Remark 3.7. Fix any R-module N. Since the resolution of any finitely generated module M over a hypersurface is eventually 2-periodic, for large enough values of i there are only two modules $\mathrm{Tor}_i(M,N)$ and $\mathrm{Ext}^i_R(M,N)$: the even and odd ones. Indeed,

$$\mathrm{Ext}^i_R(M,N) \cong \mathrm{Ext}^{i+2}_R(M,N) \quad \text{and} \quad \mathrm{Tor}^R_i(M,N) \cong \mathrm{Tor}^R_{i+2}(M,N) \quad \text{for all } i \gg 0.$$

The proof of Lemma 3.6 says that the homology of the complex P computes the even and odd *stable Tor*:

$$\mathrm{H}_{\mathrm{even}}(P) = \mathrm{Tor}^R_{2i}(M,k) \text{ for } i \gg 0 \quad \text{and} \quad \mathrm{H}_{\mathrm{odd}}(P) = \mathrm{Tor}^R_{2i+1}(M,k) \text{ for } i \gg 0.$$

Similarly, the complex P^* computes *stable Ext*:

$$\mathrm{H}_{\mathrm{even}}(P^*) = \mathrm{Ext}^{2i}_R(M,k) \text{ for } i \gg 0 \quad \text{and} \quad \mathrm{H}_{\mathrm{odd}}(P^*) = \mathrm{Ext}^{2i+1}_R(M,k) \text{ for } i \gg 0.$$

Remark 3.8. Since Q is regular, we can always choose a finite resolution F or M when constructing the complex P in Lemma 3.6. Suppose that F has length d. After fixing basis for $F_1, \ldots, F_d$, our complex P looks like

$$\cdots \longrightarrow P_{\mathrm{even}} \xrightarrow{A} P_{\mathrm{odd}} \xrightarrow{B} P_{\mathrm{even}} \xrightarrow{A} \cdots.$$

Since P is a complex,

$$\operatorname{rank} A \leqslant \operatorname{rank} \ker B \qquad \text{and} \qquad \operatorname{rank} B \leqslant \operatorname{rank} \ker A$$

and P is exact if and only if equality holds for both. Note however that by the Rank–Nulity Theorem, we only need to check equality for one: if P_{odd} has rank N, then

$$\operatorname{rank} A = \operatorname{rank} \ker B \iff N - \operatorname{rank} \ker A = N - \operatorname{rank} B \iff \operatorname{rank} B = \operatorname{rank} \ker A.$$

These ideas were extended by Eisenbud to any complete intersection.

Definition 3.9 (System of higher homotopies: general definition)**.** Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let F be a free resolution for the finitely generated R-module M, not necessarily finite. Let $\underline{f} = f_1, \ldots, f_n \in \mathrm{ann}_R(M)$. Given an n-tuple $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{Z}^n$, set $|w| := \omega_1 + \cdots + \omega_n$. A **system of higher homotopies** for $\underline{f}$ on F is a collection of R-linear maps

$$\sigma_\omega \colon F_\bullet \longrightarrow F_{\bullet+2|\omega|-1}$$

where $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{Z}^n$ with $\omega_i \geqslant 0$ for all i and such that σ_ω has degree $2|w| - 1$, satisfying the following conditions:

$$\sigma_{\mathbf{0}} = \partial_F \qquad \sigma_{\mathbf{e}_i}\sigma_{\mathbf{0}} + \sigma_{\mathbf{0}}\sigma_{\mathbf{e}_i} = f_i \cdot \mathrm{id}_F \qquad \sum_{u+v=\omega} \sigma_u \sigma_v = 0 \quad \text{for all } |\omega| \geqslant 2.$$

Here $\mathbf{0}$ denotes the n-vector with all entries 0, and $\mathbf{e}_i$ the vector with ith coordinate 1 and all other coordinates 0.

Exercise 22. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let F be a free resolution for the finitely generated R-module M, not necessarily finite. Let $\underline{f} = f_1, \ldots, f_n \in \mathrm{ann}_R(M)$. Show that there exists a system of higher homotopies for $\underline{f}$ on F.

If $\{\sigma_\omega\}$ is a system of higher homotopies for $\underline{f}$ on F, then the collection $\{\sigma_{n\mathbf{e}_i}\}$ is a system of higher homotopies for f_i on F. In fact, one can do more:

Exercise 23. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let F be a free resolution for the finitely generated R-module M, not necessarily finite. Let $\{\sigma_\omega\}$ is a system of higher homotopies for $\underline{f} = f_1, \ldots, f_n$ on F. Show that for all $a_1, \ldots, a_n \in R$ not all zero, the maps

$$\sigma_i := \sum_{|\omega|=i} a_1^{\omega_1} \cdots a_n^{\omega_n} \sigma_\omega$$

form a system of higher homotopies for $a_1 f_1 + \cdots + a_n f_n$ on F.

Remark 3.10. Note that following the system of higher homotopies for f_a from Exercise 23, the differential of the 2-periodic complex P from Lemma 3.6 becomes

$$\partial(u \otimes 1) = \sum_\omega \sigma_\omega(u) \otimes a_1^{\omega_1} \cdots a_n^{\omega_n}.$$

Theorem 3.11 (Eisenbud, 1980 [Eis80])**.** *Let Q be a regular local ring, $\underline{f} = f_1, \ldots, f_c$ a regular sequence on Q, and $R = Q/(\underline{f})$. Let M be an R-module. Given a free resolution F for M over Q and a system of higher homotopies $\{\sigma_\omega\}$ for $\underline{f}$ on F, one can construct a free resolution for M over R, as follows:*

Consider symbols $x_1^{(i)}, \ldots, x_c^{(i)}$ for all integers i and set $x_j^{(0)} = 1$ and $x_j^{(i)} = 0$ whenever $i < 0$. The complex

$$\cdots \longrightarrow \underset{n}{\bigoplus_{\substack{i_1+\cdots+i_n=d \\ d\geqslant 0}} R x_1^{(i_1)} \cdots x_c^{(i_c)} \otimes_Q F_{n-2d}} \xrightarrow{\partial} \underset{n-1}{\bigoplus_{\substack{i_1+\cdots+i_n=d \\ d\geqslant 0}} R x_1^{(i_1)} \cdots x_c^{(i_c)} \otimes_Q F_{n-1-2i}}$$

with differential

$$\partial\left(x_1^{(i_1)} \cdots x_c^{(i_c)} \otimes u\right) = \sum_\omega x_1^{(i_1-\omega_1)} \cdots x_c^{(i_c-\omega_c)} \otimes \sigma_\omega(u).$$

is a free resolution for M over R.

We end with a useful note relating systems of higher homotopies with DG algebras.

Exercise 24. Let Q be a regular local ring and $R = Q/I$ with I minimally generated by $\underline{f} = f_1, \ldots, f_n$. Let F be a free resolution of R over Q that has a structure of a DG algebra. Fix $e_1, \ldots, e_n \in F_1$ with $\partial(e_i) = f_i$. Show that we get a system of higher homotopies $\{\sigma_\omega\}$ for $\underline{f}$ on F by setting

$$\sigma_{\mathbf{e}_i}(-) = e_i \cdot - \qquad \text{and} \qquad \sigma_\omega(u) = 0 \text{ for all } |\omega| \geqslant 2.$$

4 Cohomological support varieties

Fix a noetherian local ring R. We will associate to each finitely generated R-module a variety, called the cohomological support variety of R, that encodes homological information about M. Cohomological support varieties were first defined and studied by Avramov in the 1980s, inspired by work of Quillen [Qui71]. Definition 4.2 first appeared in full generality in work of Jorgensen [Jor02], and the general theory was developed in work of Pollitz [Pol19, Pol21].

Remark 4.1. Let R be a noetherian local ring with minimal Cohen presentation $\widehat{R} \cong Q/I$ for some regular local ring $(Q, \mathfrak{m}, k)$. Note that $I/\mathfrak{m}I$ is a k-vector space of dimension $n = \mu(I)$, which we will identify with $\mathbb{A}_k^n$. A choice of coordinates for $\mathbb{A}_k^n$ corresponds to a choice of basis for k^n, and thus to a choice of minimal generating set $\underline{f} = f_1, \ldots, f_n$ for I. We will write $[f] := f + \mathfrak{m}I$ for the class of $f \in I$ in $I/\mathfrak{m}I$.

Any R-module M is also a module over $Q/(f)$ for any $f \in I$, since $fM = 0$.

Definition 4.2 (Cohomological support varieties)**.** Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let M be a finitely generated R-module. Let $\widehat{R} \cong Q/I$ be a minimal Cohen presentation. The **cohomological support variety** of M is given by

$$\mathrm{V}_R(M) := \{[f] \in I/\mathfrak{m}I \mid [f] = 0 \text{ or } \operatorname{pdim}_{Q/(f)}(\widehat{M}) = \infty\}.$$

It is not clear from the definition above that this is in fact a variety; we will prove this later in Theorem 4.10. It is also not clear that this is well-defined, meaning that it does not depend on the choice of representative f for $[f]$, nor that the definition does not depend on the choice of minimal Cohen presentation, but we will skip such details.

One can extend the definition more generally to complexes; in fact, one can talk about cohomological support varieties of elements of the bounded derived category of R. We will however not discuss such level of generality in these lectures.

Remark 4.3. Suppose that $[f]$ and $[g]$ are two points on the same line through the origin, but not the origin, so we can assume that $g = \lambda f$ for some unit $\lambda \in Q$. Then $(f) = (g)$, so $[f] \in \mathrm{V}_R(M)$ if and only if $[g] \in \mathrm{V}_R(M)$. Note moreover that $[0] \in \mathrm{V}_R(M)$ by definition. This shows that $\mathrm{V}_R(M)$ is a union of lines through the origin. Adding to this the fact that $\mathrm{V}_R(M)$ is an affine variety, we conclude that it is in fact the affine cone of a projective variety. However, there are advantages to considering $\mathrm{V}_R(M)$ as an affine variety instead of the appropriate projective version, which will unfortunately not be evident in these lectures.

Remark 4.4. Fix a minimal generating set $\underline{f} = f_1, \ldots, f_n$ for I. For each $a \in k$, write $\tilde{a}$ for a lift of a to Q. For each $a \in \mathbb{A}_k^n$, let $f_a := \widetilde{a_1} f_1 + \cdots + \widetilde{a_n} f_n$, where $\widetilde{a_i}$ for a lift of a_i to Q. We can rewrite the definition of $\mathrm{V}_R(M)$ as

$$\mathrm{V}_R(M) = \{a \in \mathbb{A}_k^n \mid a = \mathbf{0} \text{ or } \operatorname{pdim}_{Q/(f_a)}(\widehat{M}) = \infty\}.$$

Before we prove that $\mathrm{V}_R(M)$ is in fact a variety, and talk about how to compute it, let us look at some examples.

Example 4.5. Let R be a noetherian local ring with minimal Cohen presentation $\widehat{R} \cong Q/I$, where $(Q, \mathfrak{m}, k)$ is a regular local ring and $I \subseteq \mathfrak{m}^2$. Assume $n := \mu(I) \geqslant 1$. For every minimal generator $f \in I \smallsetminus \mathfrak{m}I$, by Exercise 14 the ring $Q/(f)$ is not regular, and thus $\operatorname{pdim}_{Q/(f)} k = \infty$ by Auslander–Buchsbaum–Serre (Theorem 1.11). We conclude that

$$\mathrm{V}_R(k) = \mathbb{A}_k^n.$$

Definition 4.6. Let R be a noetherian local ring with minimal Cohen presentation $\widehat{R} \cong Q/I$, where $(Q, \mathfrak{m}, k)$ is a regular local ring and $I \subseteq \mathfrak{m}^2$. Let $n := \mu(I)$. Whenever $\mathrm{V}_R(M) = \mathbb{A}_k^n$, we say that M has **full support**.

Example 4.7. Suppose that R is a complete intersection. Any minimal generator $f \in I \smallsetminus \mathfrak{m}I$ can be completed to a minimal generating set $\underline{f} = f, f_2, \ldots, f_n$ for I, and $\underline{f}$ is necessarily a regular sequence. In particular, the sequence $f_2, \ldots, f_n$ is regular on $Q/(f)$, so when we view $\widehat{R}$ as a module over $Q/(f)$, the Koszul complex on $f_2, \ldots, f_n$ is a minimal free resolution for R. In particular, $\operatorname{pdim}_{Q/(f)} \widehat{R} < \infty$, and we conclude that $\mathrm{V}_R(R) = \{\mathbf{0}\}$.

In fact, this characterizes complete intersections, though the converse is a deep theorem.

Theorem 4.8 (Pollitz, 2019 [Pol19])**.** *A noetherian local ring R is a complete intersection if and only if $\mathrm{V}_R(R) = \{\mathbf{0}\}$.*

Pollitz then used this characterization to answer a question of Dwyer, Greenlees, and Iyengar about the structure of the derived category of a noetherian local ring [DGI06]. He also used this theorem to give a new conceptual proof of the Localization Problem for complete intersections [Pol19].

We will now show that cohomological support varieties are indeed varieties. We will in fact provide an algorithm for computing $\mathrm{V}_R(M)$ for any R-module M. The theorem below is [AB00, Theorem 3.2] when R is a complete intersection, but it holds in full generality. First, some notation:

Definition 4.9. Let M be matrix with entries in a ring R. We write $I_t(M)$ for the ideal of R generated by all the t-minors of M.

Before we state the theorem, we record a useful fact:

Exercise 25. Let $(Q, \mathfrak{m})$ be a regular local ring, $I \subseteq \mathfrak{m}$ a nonzero ideal of Q, $R = Q/I$, and let M be a finitely generated R-module. Show that for any finite free resolution F for M over Q,

$$\sum_{i \geqslant 0} \operatorname{rank} F_{2i} = \sum_{i \geqslant 0} \operatorname{rank} F_{2i+1}.$$

Theorem 4.10. *Let R is be a noetherian local ring, and let $\widehat{R} \cong Q/I$ be a minimal Cohen presentation for R, meaning that $(Q, \mathfrak{m}, k)$ is a regular local ring and $I \subseteq \mathfrak{m}^2$. Suppose I is minimally generated by $\underline{f} = f_1, \dots, f_n$. For any R-module M, the cohomological support $\mathrm{V}_R(M)$ is indeed a variety. In fact we can describe this variety explicitly:*

Fix a finite free resolution F for $\widehat{M}$ over Q with $F_i = 0$ for all $i \geqslant 2d+1$, and a system of higher homotopies $\{\sigma_\omega\}$ for $\underline{f}$ on F. Consider a polynomial ring $\mathcal{S} = k[\chi_1, \dots, \chi_n]$ in n variables. Fix bases for each $F_i^* := \mathrm{Hom}_Q(F_i, k)$, and let $\sigma_\omega^{(i)}$ be the matrix representing $\mathrm{Hom}_Q(\sigma_\omega, k)\colon \mathrm{Hom}_Q(F_i, k) \longrightarrow \mathrm{Hom}_Q(F_{i+2|\omega|-1}, k)$ in the chosen bases. Finally, set

$$\sigma_i^j := \sum_{2|\omega|-1=j-i} \chi_1^{\omega_1} \cdots \chi_n^{\omega_n} \sigma_\omega^{(i)}$$

for all $i = 0, \dots, 2d$. Let $N := \displaystyle\sum_{i=0}^{d} \operatorname{rank} F_{2i}$, and consider the following $(2N) \times (2N)$ matrix:

$$C = \begin{array}{c|cccccc}
 & F_0^* & F_1^* & F_2^* & \cdots & F_{2d-1}^* & F_{2d}^* \\
\hline
F_0^* & 0 & \sigma_1^0 & 0 & \cdots & 0 & 0 \\
F_1^* & \sigma_0^1 & 0 & \sigma_2^1 & & 0 & 0 \\
F_2^* & 0 & \sigma_1^2 & 0 & & 0 & 0 \\
\cdots & & \vdots & & \ddots & \vdots & \vdots \\
F_{2d-2}^* & 0 & \sigma_1^{2d-2} & 0 & \cdots & \sigma_{2d-1}^{2d-2} & 0 \\
F_{2d-1}^* & \sigma_0^{2d-1} & 0 & \sigma_2^{2d-1} & \cdots & 0 & \sigma_{2d}^{2d-1} \\
F_{2d}^* & 0 & \sigma_1^{2d} & 0 & \cdots & \sigma_{2d-1}^{2d} & 0
\end{array}.$$

Then $\mathrm{V}_R(M)$ is the variety defined by the vanishing of the ideal $I_N(C)$ of N-minors of C.

Proof. First, consider the following two matrices:

$$\begin{array}{c|ccccc}
 & F_0^* & F_2^* & F_4^* & \cdots & F_{2d}^* \\
\hline
F_1^* & \sigma_0^1 & \sigma_2^1 & 0 & \cdots & 0 \\
F_3^* & \sigma_0^3 & \sigma_2^1 & \sigma_4^3 & & 0 \\
\cdots & \vdots & & & \ddots & \vdots \\
F_{2d-3}^* & \sigma_0^{2d-3} & \sigma_2^{2d-3} & \sigma_4^{2d-3} & \cdots & 0 \\
F_{2d-1}^* & \sigma_0^{2d-1} & \sigma_2^{2d-1} & \sigma_4^{2d-3} & \cdots & \sigma_{2d}^{2d-1}
\end{array}
\qquad
\begin{array}{c|ccccc}
 & F_1^* & F_3^* & F_5^* & \cdots & F_{2d-1}^* \\
\hline
F_1^* & \sigma_0^1 & \sigma_2^1 & 0 & \cdots & 0 \\
F_3^* & \sigma_0^3 & \sigma_2^1 & \sigma_4^3 & & 0 \\
\cdots & \vdots & & & \ddots & \vdots \\
F_{2d-3}^* & \sigma_0^{2d-3} & \sigma_2^{2d-3} & \sigma_4^{2d-3} & \cdots & 0 \\
F_{2d-1}^* & \sigma_0^{2d-1} & \sigma_2^{2d-1} & \sigma_4^{2d-3} & \cdots & \sigma_{2d}^{2d-1}
\end{array}$$

$A(\underline{\chi})$ $\qquad$ $B(\underline{\chi})$

Fix $\mathbf{0} \neq a \in \mathbb{A}_k^n$, and $f_a := \widetilde{a}_1 f_1 + \cdots \widetilde{a}_n f_n$, where $\widetilde{a}_i$ is a lift of a_i to Q. By Exercise 23, the maps

$$\sigma_i := \sum_{|\omega|=i} a_1^{\omega_1} \cdots a_n^{\omega_n} \sigma_\omega$$

are a system of higher homotopies for f_a on F.

Moreover, by Lemma 3.6, $\mathrm{pdim}_{Q/(f_a)}(\widehat{M}) < \infty$ if and only if the 2-periodic complex

$$\mathcal{P}_a^* := \cdots \longrightarrow \bigoplus_{i\geqslant 0} \mathrm{Hom}_Q(F_{2i}, k) \xrightarrow{\partial} \bigoplus_{i\geqslant 0} \mathrm{Hom}_Q(F_{2i+1}, k) \xrightarrow{\partial} \bigoplus_{i\geqslant 0} \mathrm{Hom}_Q(F_{2i}, k) \longrightarrow \cdots$$

with differential

$$\partial = \mathrm{Hom}_Q\left(\sum_{\omega} a_1^{\omega_1}\cdots a_n^{\omega_n}\sigma_\omega, k\right)$$

is exact. Thus $a \in \mathrm{V}_R(M)$ if and only if $\mathcal{P}_a^*$ is not exact. To simplify the notation, write

$$\mathcal{P}_a^{\mathrm{even}} := \bigoplus_{i\geqslant 0} \mathrm{Hom}_Q(F_{2i}, k) \qquad \text{and} \qquad \mathcal{P}_a^{\mathrm{odd}} := \bigoplus_{i\geqslant 0} \mathrm{Hom}_Q(F_{2i+1}, k).$$

Note that $\mathcal{P}_a^*$ is in fact given by

$$\mathcal{P}_a^* = \cdots \xrightarrow{B(a)} \mathcal{P}_a^{\mathrm{even}} \xrightarrow{A(a)} \mathcal{P}_a^{\mathrm{odd}} \xrightarrow{B(a)} \mathcal{P}_a^{\mathrm{even}} \xrightarrow{A(a)} \cdots$$

where $A(a)$ and $B(a)$ are obtained by setting $\chi_i = a_i$ in $A(\underline{\chi})$ and $B(\underline{\chi})$.

Now let us go back to the matrix C in the statement, and note that after reordering the rows and columns

$$C = \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}.$$

Note moreover that

$$C^2 = \begin{bmatrix} BA & 0 \\ 0 & AB \end{bmatrix} = 0,$$

and thus nonexactness of $\mathcal{P}_a^*$ is equivalent to nonexactness of

$$\cdots \xrightarrow{C(a)} \mathcal{P}_{\mathrm{even}} \oplus P_{\mathrm{odd}} \xrightarrow{C(a)} P_{\mathrm{even}} \oplus P_{\mathrm{odd}} \xrightarrow{C(a)} P_{\mathrm{even}} \oplus P_{\mathrm{odd}} \xrightarrow{C(a)} \cdots$$

or equivalently of the differential module determined by $C(a)$.

Any matrix C of size $(2N)\times(2N)$ satisfying $C^2 = 0$ has rank at most N, and C determines an exact complex precisely when it has rank N. Thus

$$a \in \mathrm{V}_R(M) \iff \mathcal{P}_a^* \text{ is not exact } \iff \operatorname{rank} C(a) < N.$$

The space of such a is determined by the vanishing of the $N \times N$ minors of C. $\square$

Remark 4.11. The proof of Theorem 4.10 gives us an alternative way to think of $\mathrm{V}_R(M)$, as the support of the 2-periodic complex corresponding to the matrices

$$\begin{array}{c|ccccc} & F_0^* & F_2^* & F_4^* & \cdots & F_{2d}^* \\ \hline F_1^* & \sigma_0^1 & \sigma_2^1 & 0 & \cdots & 0 \\ F_3^* & \sigma_0^3 & \sigma_2^1 & \sigma_4^3 & & 0 \\ \cdots & \vdots & & & \ddots & \vdots \\ F_{2d-3}^* & \sigma_0^{2d-3} & \sigma_2^{2d-3} & \sigma_4^{2d-3} & \cdots & 0 \\ F_{2d-1}^* & \sigma_0^{2d-1} & \sigma_2^{2d-1} & \sigma_4^{2d-3} & \cdots & \sigma_{2d}^{2d-1} \end{array} \qquad \begin{array}{c|ccccc} & F_1^* & F_3^* & F_5^* & \cdots & F_{2d-1}^* \\ \hline F_1^* & \sigma_0^1 & \sigma_2^1 & 0 & \cdots & 0 \\ F_3^* & \sigma_0^3 & \sigma_2^1 & \sigma_4^3 & & 0 \\ \cdots & \vdots & & & \ddots & \vdots \\ F_{2d-3}^* & \sigma_0^{2d-3} & \sigma_2^{2d-3} & \sigma_4^{2d-3} & \cdots & 0 \\ F_{2d-1}^* & \sigma_0^{2d-1} & \sigma_2^{2d-1} & \sigma_4^{2d-3} & \cdots & \sigma_{2d}^{2d-1} \end{array}$$

$$A(\underline{\chi}) \qquad\qquad\qquad\qquad B(\underline{\chi})$$

We now take Theorem 4.10 as a recipe for computing $\mathrm{V}_R(M)$.

Example 4.12. Let $Q = k[\![x, y]\!]$, $I = (x^2, xy)$, and $R = Q/I$, and let us find $\mathrm{V}_R(R)$. From Example 1.8, Example 3.3, and Exercise 20, we take

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r, "\begin{bmatrix} y \\ -x \end{bmatrix}"] \arrow[d, dashed, "\cdot x^2"'] & Q^2 \arrow[r, "\begin{bmatrix} x^2 & xy \end{bmatrix}"] \arrow[d, dashed, "\cdot x^2"'] \arrow[dl, "\begin{bmatrix} 0 & -x \end{bmatrix}"'] & Q \arrow[r] \arrow[d, dashed, "\cdot x^2"] \arrow[dl, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"'] & 0 \\
0 \arrow[r] & Q \arrow[r] & Q^2 \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

$\sigma_{(1,0)}$

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r, "\begin{bmatrix} y \\ -x \end{bmatrix}"] \arrow[d, dashed, "\cdot x^2"'] & Q^2 \arrow[r, "\begin{bmatrix} x^2 & xy \end{bmatrix}"] \arrow[d, dashed, "\cdot x^2"'] \arrow[dl, "\begin{bmatrix} x & 0 \end{bmatrix}"'] & Q \arrow[r] \arrow[d, dashed, "\cdot x^2"] \arrow[dl, "\begin{bmatrix} 0 \\ 1 \end{bmatrix}"'] & 0 \\
0 \arrow[r] & Q \arrow[r] & Q^2 \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

$\sigma_{(0,1)}$

Since our resolution has length 2, there are no other higher homotopies.

Now let us write the 2-periodic complex $\mathcal{P}$ we described in Theorem 4.10. Write e_1, e_2 for the two basis elements in F_1 with $\partial(e_1) = x^2$ and $\partial(e_2) = xy$, and let v be the basis element for F_2, so that $\partial(v) = ye_1 - xe_2$. Note that when we tensor down to k, we keep only units. In particular, since we picked F to be minimal, the differential will disappear, and only the units in our homotopies will survive.

Using the notation of the proof of Theorem 4.10, we have $\mathcal{P}_{\text{even}} = k^2$ and $\mathcal{P}_{\text{odd}} = k^2$, and our two generic matrices are

$$\begin{array}{c|cc} & e_1 & e_2 \\ \hline 1 & 0 & 0 \\ v & 0 & 0 \end{array} \qquad \begin{array}{c|cc} & 1 & v \\ \hline e_1 & \chi_1 & 0 \\ e_2 & \chi_2 & 0 \end{array}$$

In this case, it is clear that the complex is *never* exact, and thus $\mathrm{V}_R(R)$ has full support.

Alternatively, one might write the matrix C from the statement of Theorem 4.10:

$$C = \begin{array}{c|cccc} & 1 & e_1 & e_2 & v \\ \hline 1 & 0 & 0 & 0 & 0 \\ e_1 & \chi_1 & 0 & 0 & 0 \\ e_2 & \chi_2 & 0 & 0 & 0 \\ v & 0 & 0 & 0 & 0 \end{array}$$

and note that C has rank 1, so its 2×2 minors vanish. Thus $\mathrm{V}_R(M)$ is the variety corresponding to (0), and M has full support.

The following exercise will soon be helpful:

Exercise 26. Let $(Q, \mathfrak{m})$ be a regular local ring and F be a complex of finitely generated free Q-modules, not necessarily bounded on either side. Show that F is exact if and only if $F \otimes_Q Q/\mathfrak{m}$ is exact.

We will now describe cohomological support varieties over complete intersections in a more compact format, which will in fact recover Avramov's original definition:

Remark 4.13. Let k be an algebraically closed field. Let $R = Q/(\underline{f})$ where $(Q, \mathfrak{m}, k)$ is a regular local ring and $\underline{f} = f_1, \ldots, f_c$ is a regular sequence. Consider the polynomial ring $\mathcal{S} = k[\chi_1, \ldots, \chi_c]$, which we give the grading where each variable χ_i has degree -2.[5] This ring $\mathcal{S}$ is known as the **ring of cohomological operators** or **ring of Eisenbud operators**. Let F be any finite free resolution of M over Q, and consider Eisenbud's recipe for a free resolution over R given in Theorem 3.11:

$$G = \cdots \longrightarrow \underbrace{\bigoplus_{\substack{i_1+\cdots+i_n=d \\ d\geqslant 0}} Rx_1^{(i_1)} \cdots x_c^{(i_c)} \otimes_Q F_{n-2d}}_{n} \xrightarrow{\partial} \underbrace{\bigoplus_{\substack{i_1+\cdots+i_n=d \\ d\geqslant 0}} Rx_1^{(i_1)} \cdots x_c^{(i_c)} \otimes_Q F_{n-1-2d}}_{n-1} \longrightarrow \cdots$$

with differential

$$\partial\left(x_1^{(i_1)} \cdots x_c^{(i_c)} \otimes u\right) = \sum_{\omega} x_1^{(i_1-\omega_1)} \cdots x_c^{(i_c-\omega_c)} \otimes \sigma_\omega(u).$$

Set $x^{(d)} = 0$ for $d < 0$ and $x^{(0)} = 1$, and define

$$\chi_i \cdot x_1^{(j_1)} \cdots x_c^{(j_c)} := x_1^{(j_1)} \cdots x_{i-1}^{(j_{i-1})} x_i^{(j_i-1)} x_{i+1}^{(j_{i+1})} \cdots x_c^{(i_c)}.$$

One can easily check that the action of χ_i and χ_j commute with each other, and $\chi_i\partial = \partial\chi_i$, so this gives G the structure of a graded module over $\mathcal{S}$.

On the other hand, the homology of the complex $\operatorname{Hom}_R(G,k)$ computes

$$\operatorname{Ext}_R^*(M,k) = \bigoplus_{i\geqslant 0} \operatorname{Ext}_R^i(M,k).$$

so we get an induced graded $\mathcal{S}$-module structure on $\operatorname{Ext}_R^*(M,k)$ with

$$\chi_i \colon \operatorname{Ext}_R^i(M,k) \longrightarrow \operatorname{Ext}_R^{i+2}(M,k).$$

Note that the graded module underlying $\operatorname{Hom}_R(G,k)$ is a finitely generated free graded $\mathcal{S}$-module, and thus $\operatorname{Ext}_R^*(M,k)$ is finitely generated over $\mathcal{S}$.

The 2-periodic complex obtained from $\operatorname{Hom}_R(G,k)$ by taking the even and odd parts

$$\cdots \longrightarrow \bigoplus_{i\geqslant 0} \operatorname{Hom}_Q(F_{2i},k) \otimes_k \mathcal{S} \longrightarrow \bigoplus_{i\geqslant 0} \operatorname{Hom}_Q(F_{2i+1},k) \otimes_k \mathcal{S} \longrightarrow \cdots$$

is a 2-periodic complex of free $\mathcal{S}$-modules, and in fact it is precisely the complex determined by the two matrices $A(\chi)$ and $B(\chi)$ from Theorem 4.10.

Moreover, the 2-periodic complex obtained from

$$\operatorname{Hom}_R(G,k) \otimes_{\mathcal{S}} \mathcal{S}/(\chi_1 - a_1, \ldots, \chi_c - a_c)$$

is the 2-periodic complex $\mathcal{P}_a^*$ given by the matrices $A(a)$ and $B(a)$ obtained from $A(\chi)$ and $B(\chi)$ by setting $\chi_i = a_i$, as in the proof of Theorem 4.10.

[5] As Avramov wisely pointed out in [Avr10], "This will not be surprising, once the χ_i's reveal their cohomological nature."

Thus $\mathrm{V}_R(M) = \{a \in \mathbb{A}_k^c \mid \mathrm{Hom}_R(G,k) \otimes_{\mathcal{S}} \mathcal{S}/(\chi_1 - a_1, \dots, \chi_c - a_c) \text{ is not exact}\}$. Fix $a \in \mathbb{A}_k^c$ and let $\mathfrak{m} = (\chi_1 - a_1, \dots, \chi_c - a_c)$. Note that

$$\mathrm{Hom}_R(G,k) \otimes_{\mathcal{S}} \mathcal{S}/\mathfrak{m} \cong \mathrm{Hom}_R(G,k)_{\mathfrak{m}} \otimes_{\mathcal{S}_{\mathfrak{m}}} \mathcal{S}_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}.$$

By Exercise 26,

$$\mathrm{Hom}_R(G,k) \otimes_{\mathcal{S}} \mathcal{S}/\mathfrak{m} \text{ is exact} \iff \mathrm{Hom}_R(G,k)_{\mathfrak{m}} \text{ is exact}.$$

Since $\mathrm{Hom}_R(G,k)$ computes $\mathrm{Ext}_R^*(M,k)$, we conclude that

$$a \notin \mathrm{V}_R(M) \iff \mathrm{Hom}_R(G,k) \otimes_{\mathcal{S}} \mathcal{S}/\mathfrak{m} \text{ is exact} \iff \mathrm{Ext}_R^*(M,k)_{\mathfrak{m}} = 0.$$

Therefore,

$$a \in \mathrm{V}_R(M) \iff (\chi_1 - a_1, \dots, \chi_c - a_c) \in \mathrm{Supp}_{\mathcal{S}} \mathrm{Ext}_R^*(M,k).$$

By Nullstellensatz, we conclude that

$$\sqrt{\mathrm{ann}_{\mathcal{S}} \mathrm{Ext}_R^*(M,k)} = \bigcap_{a \in \mathrm{V}_R(M)} (\chi_1 - a_1, \dots, \chi_c - a_c).$$

Thus the radical ideal in $k[\chi_1, \dots, \chi_n]$ defining the variety $\mathrm{V}_R(M)$ determines the support of $\mathrm{Ext}_R^*(M,k)$. This explains the words *cohomological* and *support* in the name cohomological support varieties.

It is a well-known fact from dimension theory (see, for example, [Mat89, Theorem 13.4]) that the dimension of a finitely generated graded module is the rate at which the rank of the graded pieces grow. We have shown that $\dim \mathrm{V}_R(M)$ is the dimension of the graded $\mathcal{S}$-module $\mathrm{Ext}_R^*(M,k)$. Thus when R is a complete intersection (see [AB00])

$$\mathrm{cx}_R(M) = \dim \mathrm{V}_R(M).$$

Avramov and Buchweitz used cohomological support varieties to show the following surprising fact:

Theorem 4.14 (Avramov–Buchweitz, 2000 [AB00])**.** *Let R be a local complete intersection, and let M and N be finitely generated R-modules. Then*

$$\mathrm{Ext}_R^i(M,N) = 0 \textit{ for all } i \gg 0 \iff \mathrm{Ext}_R^i(M,N) = 0 \textit{ for all } i \gg 0.$$

Their proof amounts to showing that the condition

$$\mathrm{Ext}_R^i(M,N) = 0 \text{ for all } i \gg 0$$

is equivalent to $\mathrm{V}_R(M) \cap \mathrm{V}_R(N) = \{0\}$. The theorem then follows immediately.

Remark 4.15. More generally, when R is not a complete intersection, one can still recast $\mathrm{V}_R(M)$ as the support of a certain Ext-module; this is the definition most commonly used by experts. Taking E to be the Koszul complex on a minimal generating set for I, where $R = Q/I$ and Q is a regular local ring, there is an action of the ring of cohomological operators $\mathcal{S} = k[\chi_1, \dots, \chi_n]$ on $\mathrm{Ext}_E^*(M,k)$ making it a finitely generated $\mathcal{S}$-module, and

$$\mathrm{V}_R(M) = \mathrm{Supp}_{\mathcal{S}} \mathrm{Ext}_E^*(M,k).$$

Though in this generality, $\mathrm{V}_R(M)$ no longer measures the complexity of M over R, but rather its complexity over E.

Problem 4.16. *Which subvarieties $V \subseteq \mathbb{A}_k^n$ can be realized as the cohomological support variety $\mathrm{V}_R(M) = V$ for some R-module M?*

One obvious requirement is that V needs to a **conical variety**, meaning it must be a union of lines through the origin. When R is a complete intersection, there are no other requirements. This was showed in an unpublished preprint of Avramov and Jorgensen, and independently (and via different methods) by Bergh [Ber07]. Avramov and Iyengar later gave a method for constructing modules with any prescribed support [AI07].

Theorem 4.17 (Bergh, 2007 [Ber07], Avramov–Jorgensen, Avramov–Iyengar, [AI07])**.** *Let Q be a regular local ring and I be an ideal in Q generated by a regular sequence of length n. Any conical variety $V \subseteq \mathbb{A}_k^n$ can be realized as $V = \mathrm{V}_R(M)$ for some R-module M.*

In fact, Bergh proved that one can realize any variety with a maximal Cohen-Macaulay module.

But when R is not a complete intersection, some varieties *cannot* be realized.

Definition 4.18. Let R be a noetherian local ring. The **complete intersection defect** of R, written $\mathrm{cid}(R)$, is defined as

$$\mathrm{cid}(R) := \varepsilon_2(R) - \varepsilon_1(R) + \mathrm{depth}(R).$$

Exercise 27. Let R be a noetherian local ring and let $\widehat{R} \cong Q/I$ be a minimal Cohen presentation for R. Show that

$$\mathrm{cid}(R) = \mu(I) - \mathrm{height}(I).$$

This explains the name: a ring R is a complete intersection if and only if $\mathrm{cid}(R) = 0$, and in general $\mathrm{cid}(R)$ measures how far the defining ideal I of $\widehat{R}$ is from being generated by a regular sequence. The advantage of the first definition we gave is that it does not require choosing a minimal Cohen presentation for R.

Theorem 4.19 (Briggs–G–Pollitz, 2024 [BGP24])**.** *Let R be a Cohen-Macaulay local ring. If R is not a complete intersection, then for every R-module M*

$$\dim \mathrm{V}_R(M) > \mathrm{cid}(R).$$

Note that in particular, there are no modules with trivial support $\mathrm{V}_R(M) = \{0\}$, but even more: there are no modules whose support is a line.

Theorem 4.20 (Briggs–G–Pollitz [BGP25])**.** *Let R be a noetherian local ring with minimal Cohen presentation $\widehat{R} \cong Q/I$, where $I \subseteq \mathfrak{m}^2$. If I is generated a monomials on some regular sequence $x_1, \ldots, x_d$ of R, and R is not a complete intersection, then for every R-module M*

$$\dim (\mathrm{V}_R(M)) \geqslant \mathrm{cid}(R).$$

The question of whether this holds more generally for any noetherian local ring that is not a complete intersection remains open.

Example 4.21. Consider $Q = k[\![x, y, z, w]\!]$, $I = (x^2, xy, yz, zw, w^2)$, and $R = Q/I$. This ring is not Cohen-Macaulay ring and it has complete intersection defect 2. Theorem 4.20 says that $\dim \mathrm{V}_R(M) \geqslant 2$ for all nonzero complexes M with finitely generated homology. One can easily find M with $\dim \mathrm{V}_R(M) = 3$, such as the cyclic module $M = R/(y, z)$. Indeed, one can compute directly, or apply [BGP22, Lemma 2.6], to see that the cohomological support variety of M is a 3-dimensional hyperplane. We do not know if there is an R-complex with finitely generated homology that has a 2-dimensional cohomological support variety.

Index

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