Symbolic Powers

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These are in no way comprehensive, but more about symbolic powers can be found in the references — for example, see the surveys [DDSG+18] and [SS17].

1 An introduction to Symbolic Powers

Our goal is to study symbolic powers. Besides being an interesting subject in its own right, symbolic powers appear as auxiliary tools in several important results in commutative algebra, such as Krull's Principal Ideal Theorem, Chevalley's Lemma, or in giving a proof in prime characteristic for the fact that regular local rings are UFDs. Hartshorne's proof of the Hartshorne—Lichtenbaum Vanishing Theorem also makes use of symbolic powers. Explicitly, Hartshorne's proof of this local cohomology result uses the fact that certain symbolic and -adic topologies are equivalent, and thus local cohomology can be computed using symbolic powers.

1.1 Primary decomposition and associated primes

One of the fundamental classical results in commutative algebra is the fact that every ideal in any noetherian ring has a primary decomposition. This can be thought of as a generalization of the Fundamental Theorem of Arithmetic, which says that every integer n can be written as a product of powers of primes. In fact, such a product *is* a primary decomposition: if $n = p_1^{a_1} \cdots p_k^{a_k}$, then the primary decomposition of the ideal (n) is $(n) = (p_1^{a_1}) \cap \cdots \cap (p_k^{a_k})$. However, this example can be deceiving, in that it suggests that primary ideals are just powers of primes; they are not!

Definition 1.1. A proper ideal Q in a ring R is called *primary* if the following holds for all $a, b \in R$: if $ab \in Q$, then $a \in Q$ or $b^n \in Q$ for some $n \geqslant 1$.

Remark 1.2. From the definition, it follows that the radical of a primary ideal is always a prime ideal. If the radical of a primary ideal Q is the prime ideal P, we say that Q is P-primary.

Exercise 1.3. If the radical of a I is maximal, then I is primary.

Note, however, that not all ideals with a prime radical are primary, as we will see in Example 1.23.

Definition 1.4 (Irredundant Primary Decomposition)**.** A *primary decomposition* of the ideal I consists of primary ideals $Q_1, \ldots, Q_n$ such that $I = Q_1 \cap \cdots \cap Q_n$. An *irredundant* primary decomposition of I is one such that no Q_i can be omitted, and such that $\sqrt{Q_i} \neq \sqrt{Q_j}$ for all $i \neq j$.

Exercise 1.5. Show that a finite intersection of P-primary ideals is a P-primary ideal.

Remark 1.6. Any primary decomposition can be simplified to an irredundant one. This can be achieved by deleting unnecessary components and intersecting primary ideals with the same radical, since the intersection of primary ideals with the same radical P is in fact a P-primary ideal.

As advertised, primary decompositions always exist:

Theorem 1.7 (Lasker [Las05], Noether [Noe21])**.** Every ideal in a noetherian ring has a primary decomposition.

Proof. For a modern proof, see [Mat80, Section 8]. □

Example 1.8. Here are some examples of primary decompositions:

a) If I is a radical ideal, I coincides with the intersection of its minimal primes; there are finitely many such primes as we are over in a noetherian ring. Since prime ideals are primary, writing I as the intersection of its minimal primes is precisely a primary decomposition for I.

b) The ideal (xy, xz, yz) in $\mathbb{C}[x, y, z]$ is radical, so we just need to find its minimal primes. One can check that the decomposition is $(xy, xz, yz) = (x, y) \cap (x, z) \cap (y, z)$. More generally, the radical monomial ideals are precisely those that are squarefree, and the primary components of a monomial ideal are also monomial.

c) Primary decompositions, even irredundant ones, are not unique. For example, over any field k, the ideal (x^2, xy) in $k[x, y]$ has infinitely many irredundant primary decompositions: given any $n \geqslant 1$, $(x^2, xy) = (x) \cap (x^2, xy, y^n)$. One thing all of these have in common is the radicals of the primary components: they are always (x) and (x, y).

What information can we extract from a primary decomposition? Is there any sense in which primary decompositions are unique? What primes can appear as radicals of the primary components of I? Let's start with the last question: the prime ideals that appear are indeed interesting.

Definition 1.9 (Associated Prime)**.** Let M be an R-module. A prime ideal P is an *associated prime* of M if the following equivalent conditions hold:

(a) There exists a non-zero element $a \in M$ such that $P = \operatorname{ann}_R(a)$.

(b) There is an inclusion of R/P into M.

If I is an ideal of R, we refer to an associated prime of the R-module R/I as simply an associated prime of I. We will denote the set of associated primes of I by $\operatorname{Ass}(R/I)$.

Over a noetherian ring, the set of associated primes of an ideal $I \neq 0$ is always non-empty and finite. Moreover, $\operatorname{Ass}(R/I) \subseteq \operatorname{Supp}(R/I)$, where $\operatorname{Supp}(M)$ denotes the support of the module M, meaning the set of primes p such that $M_p \neq 0$. In fact, the minimal primes of the support of R/I coincide with the minimal associated primes of I: those are precisely all the minimal primes of I. For proofs of these facts and more on associated primes, see [Mat80, Section 7].

Exercise 1.10. Let R be a noetherian ring and I an ideal in R. Show that a prime ideal P is associated to I if and only if $\operatorname{depth}(R_P/I_P) = 0$.

Given an ideal I, we will be interested not only in its associated primes, but also in the associated primes of its powers. Fortunately, the set of prime ideals that are associated to some power of I is finite, a result first proved by Ratliff [Rat76] and then extended by Brodmann [Bro79].

Definition 1.11. Let R be a noetherian domain and I a non-zero ideal in R. We define

$$A(I) = \bigcup_{n \geqslant 1} \operatorname{Ass}(R/I^n).$$

Theorem 1.12 (Brodmann, 1979)**.** Let R be a noetherian domain and $I \neq 0$ an ideal in R. For n sufficiently large, $\operatorname{Ass}(R/I^n)$ is independent of n. In particular, $A(I)$ is a finite set.

The relationship between primary decomposition and associated primes is as follows:

Theorem 1.13 (Primary Decomposition)**.** Let $I = Q_1 \cap \cdots \cap Q_n$ be an irredundant primary decomposition of I, where Q_i is a P_i-primary ideal for each i. Then

$$\operatorname{Ass}(R/I) = \{P_1, \ldots, P_n\}.$$

Moreover, if P_i is minimal in $\operatorname{Ass}(R/I)$, then Q_i is unique, and given by

$$Q_i = I_{P_i} \cap R,$$

where $- \cap R$ denotes the pre-image in R via the natural map $R \longrightarrow R_P$.

Proof. See [Mat80, Section 8]. ☐

However, if P_i is an *embedded* prime of I, meaning that P_i is not minimal in $\operatorname{Ass}(R/I)$, then the corresponding primary component is not necessarily unique.

Example 1.14. Let's look back at our last example, the ideal $I = (x^2, xy)$ in $k[x, y]$. All its irredundant primary decompositions must have precisely two components, one for each associated prime of I: (x) and (x, y). The minimal component is always (x), since when we localize at (x), y becomes invertible and $I_{(x)} = (x)_{(x)}$. The other component is (x, y)-primary, since (x, y) is the only embedded prime of I. That embedded component, as we saw before, can now take many forms, such as (x^2, xy, y^n) for any n.

1.2 Symbolic powers: definition and basic properties

The n-th symbolic power of a radical ideal I is obtained by collecting all the primary components of I^n that correspond to minimal primes, and discarding all the embedded components.

Definition 1.15 (Symbolic Powers)**.** Let R be a noetherian ring, and I an ideal in R with no embedded primes. The *n-th symbolic power* of I is the ideal defined by

$$I^{(n)} = \bigcap_{P \in \operatorname{Min}(R/I)} (I^n R_P \cap R).$$

Remark 1.16. In the case of a prime ideal P, its n-th symbolic power is given by

$$P^{(n)} = P^n R_P \cap R = \{a \in R : sa \in P^n \text{ for some } s \notin P\}.$$

Here are some properties that completely characterize the symbolic powers of a prime P:

- The n-th symbolic power of a P is the unique P-primary component in an irredundant primary decomposition of P^n.

- The n-th symbolic power of P is the smallest P-primary ideal containing P^n.

The equality $P^{(n)} = P^n$ is thus equivalent to the condition that P^n is a primary ideal. In particular, if $\mathfrak{m}$ is a maximal ideal, $\mathfrak{m}^n = \mathfrak{m}^{(n)}$ for all n; indeed, an embedded prime of $\mathfrak{m}^n$ would be a prime ideal strictly containing the only minimal prime, $\mathfrak{m}$ itself, and such a prime cannot exist.

Remark 1.17. If $I = P_1 \cap \cdots \cap P_k$ is a radical ideal with minimal primes $P_1, \ldots P_k$,

$$I^{(n)} = P_1^{(n)} \cap \cdots \cap P_k^{(n)}.$$

This is a simple consequence of the fact that $IR_P = PR_P$ for each minimal prime P.

Exercise 1.18. Show that if P is prime, $P^{(n)}$ is the smallest P-primary ideal containing P^n.

Exercise 1.19. Show that if $\mathfrak{m}$ is a maximal ideal, $\mathfrak{m}^n = \mathfrak{m}^{(n)}$ for all n.

Remark 1.20. In the definition above, the assumption that I has no embedded primes implies in particular that $\operatorname{Ass}(I) = \operatorname{Min}(I)$. However, when I has embedded primes, we do have two distinct possible definitions for symbolic powers, given by intersecting $I^n R_P \cap R$ with P ranging over $\operatorname{Ass}(I)$ or $\operatorname{Min}(I)$. We will focus on ideals with no embedded primes, so this distinction is not relevant.

Both definitions have advantages. When we take P ranging over $\operatorname{Ass}(I)$, we get $I^{(1)} = I$, while taking P ranging over $\operatorname{Min}(I)$ means that $I^{(n)}$ coincides with the intersection of the primary components of I^n corresponding to its minimal primes.

As a warning, we will assume that I has no embedded primes throughout.

Lemma 1.21. Let I be an ideal with no embedded primes in a noetherian ring R.

(a) $I^{(1)} = I$;

(b) For all $n \geqslant 1$, $I^n \subseteq I^{(n)}$;

(c) $I^a \subseteq I^{(b)}$ if and only if $a \geqslant b$.

(d) If $a \geqslant b$, then $I^{(a)} \subseteq I^{(b)}$;

(e) For all $a, b \geqslant 1$, $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$.

(f) $I^n = I^{(n)}$ if and only if I^n has no embedded primes.

Proof.

(a) Since all of the associated primes of I are minimal, Theorem 1.13 guarantees that

$$I \quad = \bigcap_{P \in \operatorname{Ass}(R/I)} (IR_P \cap R) = I^{(1)}.$$

(b) For all associated primes P of I, $I^n \subseteq I^n R_P \cap R$. In other words, any set is contained in the preimage of its own image by any map.

(c) If $a \geqslant b$, then $I^a \subseteq I^b \subseteq I^{(b)}$. Conversely, if $I^a \subseteq I^{(b)}$, then given an associated prime P of I, we must have $(I_P)^a = (I^a)_P \subseteq \left(I^{(b)}\right)_P = (I_P)^b$. Write $J = I_P$. If $a < b$, it would follow that $J^a = J^b$, which by Nakayma's Lemma implies $J = 0$, and thus $I = 0$.

(d) Follows from the fact that $I^a \subseteq I^b$, and that taking preimages preserves inclusions.

(e) The containment follows from the identity $I^a I^b = I^{a+b}$. If $x \in I^{(a)}$ and $y \in I^{(b)}$, then the natural map $R \longrightarrow R_P$ takes xy into the image of I^{a+b}.

(f) Since $\sqrt{I^n} = \sqrt{I}$, the minimal primes of I^n coincide with those of I. Therefore, an irredundant primary decomposition of I^n consists of

$$I^n = I^{(n)} \cap Q_1 \cap \cdots \cap Q_k,$$

where $Q_1, \ldots, Q_k$ are primary components corresponding to embedded primes of I^n. There are no such components precisely when $I^n = I^{(n)}$. $\square$

As (e) suggests, even if I has no embedded primes, I^n may still have some embedded primes, and in particular the converse containments to (b) and (d) do not hold in general.

Symbolic powers do coincide with ordinary powers if the ideal is generated by a regular sequence. However, this is far from being an if and only if — more on that later.

Lemma 1.22. If I is generated by a regular sequence in a Cohen-Macaulay ring, then $I^n = I^{(n)}$ for all $n \geqslant 1$.

Proof. Since I is generated by a regular sequence, the associated graded ring of I, which is the graded ring $\bigoplus_{n \geqslant 0} I^n/I^{n+1}$, is isomorphic to a polynomial ring over R/I in as many variables as generators of I. In particular, I^n/I^{n+1} is a free module over R/I for each n. Therefore, the associated primes of R/I and I^n/I^{n+1} coincide. Now consider the following exact sequence:

$$0 \longrightarrow I^n/I^{n+1} \longrightarrow R/I^{n+1} \longrightarrow R/I^n \longrightarrow 0\,.$$

By [Mat80, Lemma 7.F], $\operatorname{Ass}(R/I^{n+1}) \subseteq \operatorname{Ass}(I^n/I^{n+1}) \cup \operatorname{Ass}(R/I^n)$. When $n = 1$, this implies that $\operatorname{Ass}(R/I^2) \subseteq \operatorname{Ass}(R/I)$.

On the other hand, we have $\operatorname{Ass}(R/I) \subseteq \operatorname{Ass}(R/I^2)$ whenever I itself has no embedded primes, since $\operatorname{Min}(R/I) = \operatorname{Min}(R/I^2)$ always holds. Our assumption that I is generated by a regular sequence implies in particular that R/I is Cohen-Macaulay, and thus I has no embedded primes. We conclude that the associated primes of I^2 coincide with those of I. Now proceeding by induction, the result follows. $\square$

In particular, the symbolic powers of a prime ideal are not, in general, trivial:

Example 1.23. Consider a field k and an integer $n > 1$ and let $A = k[x, y, z]$, $p = (x, z)$, $I = (xy - z^n)$ and $R = A/I$. Consider the prime ideal $P = p/I$, and note that $y \notin P$. Since $xy = z^n \in P^n$, we have $x \in P^{(n)}$. However, $x \notin P^n$, so in fact $P^n \subsetneq P^{(n)}$.

In particular, P^n is not a primary ideal, even though its radical is the prime P.

The equality of ordinary and symbolic powers of a prime ideal might fail even over a regular ring:

Exercise 1.24. Consider the ideal $I = I_2(X)$ of 2×2 minors of a generic 3×3 matrix

$$X = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix}$$

in the polynomial ring $R = k[X] = k\left[x_{i,j} \,|\, 1 \leqslant i, j \leqslant 3\right]$ generated by the variables in X over a field k. Show that $g = \det X \in P^{(2)}$, while $g \notin P^2$.

Example 1.25. Let k be a field, $R = k[x, y, z]$, and consider the map $\psi : R \longrightarrow k[t]$ given by $\psi(x) = t^3$, $\psi(y) = t^4$ and $\psi(z) = t^5$. Let P be the prime ideal

$$P = \ker \psi = \left(\underbrace{x^2y - z^2}_{f}, \underbrace{xz - y^2}_{g}, \underbrace{yz - x^3}_{h} \right).$$

We will see that $P^{(2)} \neq P^2$.

First, consider a non-standard grading on R under which ψ is a degree 0 map, and I is a homogeneous ideal: give x degree 3, y degree 4, and z degree 5. With this grading, $\deg(f) = 10$, $\deg(g) = 8$ and $\deg(h) = 9$, and the polynomial $fg - h^2$ is homogeneous of degree 18. Note that $fg - h^2 = xq$, for some q of degree $18 - 3 = 15$. Since $x \notin P$ and $fg - h^2 \in P^2$, we have $q \in P^{(2)}$. However, since all elements in P have degree at least 8, then all elements in P^2 must have degree at least 16, so that $q \notin P^2$. We conclude that $P^2 \neq P^{(2)}$.

1.3 Symbolic powers and geometry

One motivation to study symbolic powers is that over a regular ring they correspond to a natural geometric notion of power, by the following classical result:

Theorem 1.26 (Zariski–Nagata [Zar49, Nag62])**.** Let $R = \mathbb{C}[x_1, \ldots, x_d]$ and let I be a radical ideal in R. Then for all $n \geqslant 1$,

$$I^{(n)} = \bigcap_{\substack{\mathfrak{m} \supseteq I \\ \mathfrak{m} \in \mathrm{mSpec}(R)}} \mathfrak{m}^n.$$

We can think of this as a higher order version of Hilbert's Nullstellensatz. The Nullstellensatz tells us that the maximal ideals in R are in bijection with the points in $\mathbb{C}^d$, and that the polynomials in our radical ideal I are those that vanish on each point of the corresponding algebraic variety in k^d:

$$I = \bigcap_{\substack{\mathfrak{m} \supseteq I \\ \mathfrak{m} \in \mathrm{mSpec}(R)}} \mathfrak{m}.$$

The polynomials that vanish at each particular point on the variety corresponding to I form a maximal ideal $\mathfrak{m}$ that contains I. Zariski–Nagata then says that the polynomials that vanish to order n along that variety are precisely those in $I^{(n)}$ — the polynomials in $\mathfrak{m}^n$ are those that vanish to order n at the point that corresponds to $\mathfrak{m}$.

We will prove this theorem by providing an additional description, in terms of differential operators:

$$I^{(n)} = \left\{ f \in R \,\middle|\, \frac{\partial^{a_1+\cdots+a_d}}{\partial x_1^{a_1}\cdots x_d^{a_d}}(f) \in I \text{ for all } a_1+\cdots+a_d < n \right\} = \bigcap_{\substack{\mathfrak{m}\supseteq I \\ \mathfrak{m}\in \operatorname{mSpec}(R)}} \mathfrak{m}^n.$$

We can describe symbolic powers in terms of differential operators in greater generality, over any perfect field. For that, we will need the following definition:

Definition 1.27 (Differential operators)**.** Given a finitely generated k-algebra R, the k-linear differential operators of R of order n, $D_R^n \subseteq \operatorname{Hom}_k(R,R)$, are defined as follows:

- The differential operators of order zero are simply the R-linear maps:
$$D_R^0 = R \cong \operatorname{Hom}_R(R,R).$$
- We say that $\delta \in \operatorname{Hom}_k(R,R)$ is an operator of order up to n, meaning $\delta \in D_R^n$, if
$$[\delta, r] = \delta r - r\delta$$
is an operator of order up to $n-1$ for all $r \in D_R^0$.

The ring of k-linear differential operators is defined by $D_R = \bigcup_{n\in\mathbb{N}} D_R^n$.

If R is clear from the context, we drop the subscript referring to the ring.

Definition 1.28. Let R be a finitely generated k-algebra, I an ideal of R, and n be a positive integer. The n-th k-linear differential power of I is given by

$$I^{\langle n\rangle} = \{ f \in R \mid \delta(f) \in I \text{ for all } \delta \in D_R^{n-1} \}.$$

The Zariski–Nagata Theorem will then say that the differential powers of I coincide with its symbolic powers:

Theorem 1.29 (Zariski–Nagata [Zar49, Nag62], see also [DDSG+18])**.** Let $R = k[x_1,\ldots,x_d]$, where k is a perfect field, and let I be a radical ideal. For all $n \geqslant 1$,

$$I^{(n)} = I^{\langle n\rangle} = \bigcap_{\substack{\mathfrak{m}\supseteq I \\ \mathfrak{m}\in \operatorname{mSpec}(R)}} \mathfrak{m}^n.$$

In order to prove this theorem, we will need to understand differential powers a little better, to ultimately show they have the properties that characterize symbolic powers. Here's what we will show for any prime ideal P:

1) $P^{\langle n\rangle}$ is a P-primary ideal. (Lemma 1.33 and Proposition 1.34)

2) $P^n \subseteq P^{\langle n\rangle}$. (Proposition 1.35)

3) $\left(P^{\langle n\rangle}\right)_P = (P_P)^{\langle n\rangle}$. (Lemma 1.36)

4) For a maximal ideal $\mathfrak{m}$, $\mathfrak{m}^n = \mathfrak{m}^{\langle n\rangle}$. (Remark 1.37)

Once we have these, 1) and 2) together imply $P^{(n)} \subseteq P^{\langle n\rangle}$, since $P^{(n)}$ is the smallest P-primary ideal containing P^n. To show $P^{\langle n\rangle} \subseteq P^{(n)}$, we only need to show the containment holds after localizing at P, which is the only associated prime of $P^{(n)}$, by Exercise 1.30 below. But 3) says differential powers commute with localization, and after localization P becomes the maximal ideal; so 4) completes the proof that $P^{\langle n\rangle} \subseteq P^{(n)}$ for a prime ideal P.

Now if we take any radical ideal I, we can write I as the intersection of finitely many primes, say

$$I = P_1 \cap \cdots \cap P_r.$$

Then

$$I^{(n)} = P_1^{(n)} \cap \cdots \cap P_r^{(n)} = P_1^{\langle n\rangle} \cap \cdots \cap P_r^{\langle n\rangle} = (P_1 \cap \cdots \cap P_r)^{\langle n\rangle} = I^{\langle n\rangle}.$$

Exercise 1.30. (Containments are local statements) Given ideals I and J in a noetherian ring R, the following are equivalent:

(a) $I \subseteq J$;

(b) $I_P \subseteq J_P$ for all primes $P \in \operatorname{Supp}(R/J)$;

(c) $I_P \subseteq J_P$ for all primes $P \in \operatorname{Ass}(R/J)$.

Exercise 1.31. Let $\{I_\alpha\}_{\alpha\in A}$ be an indexed family of ideals. Then,

$$\bigcap_{\alpha\in A} I_\alpha^{\langle n\rangle} = \left(\bigcap_{\alpha\in A} I_\alpha\right)^{\langle n\rangle}$$

for every $n \geqslant 0$.

We note that in what follows and up to Proposition 1.35, k can be any ring.

Remark 1.32. Since $D_R^{n-1} \subseteq D_R^n$, it follows that $I^{\langle n+1\rangle} \subseteq I^{\langle n\rangle}$. In fact, given any ideals $I \subseteq J$, we have $I^{\langle n\rangle} \subseteq J^{\langle n\rangle}$ for every $n \geqslant 0$.

Lemma 1.33. Let R be a finitely generated k-algebra, I be an ideal of R, and n be a positive integer. The set $I^{\langle n\rangle}$ is an ideal.

Proof. If $f, g \in I^{\langle n\rangle}$ then $f + g \in I^{\langle n\rangle}$, since for any $\delta \in D_R^{n-1}$,

$$\delta(f+g) = \underbrace{\delta(f)}_{\in I} + \underbrace{\delta(g)}_{\in I} \in I.$$

Now we need to show that $rf \in I^{\langle n\rangle}$ for any $r \in R$ and $f \in I^{\langle n\rangle}$. For any $\delta \in D^{n-1}$, note that $f \in I^{\langle n\rangle} \subseteq I^{\langle n-1\rangle}$, $[\delta, r] \in D^{n-2}$, and $\delta(f) \in I$, so

$$\delta(rf) = \underbrace{\underbrace{[\delta, r]}_{\in D^{n-2}}(\underbrace{f}_{\in I^{\langle n-1\rangle}})}_{\in I} + r\underbrace{\delta(f)}_{\in I} \in I.$$

We conclude that $\delta(rf) \in I$. Hence, $rf \in I^{\langle n\rangle}$. □

Proposition 1.34. Let R be a finitely generated k-algebra. Let P be a prime ideal of R, and n be a positive integer. Then, $P^{\langle n\rangle}$ is P-primary.

Proof. We use induction on n. The base case is clear: $P^{\langle 1\rangle} = P$ is clearly P-primary.

Now suppose that $P^{\langle n\rangle}$ is P-primary. To show that $P^{\langle n+1\rangle}$ is also P-primary, we need to show that whenever $r \notin P$ and $f \in P$ are such that $rf \in P^{\langle n+1\rangle}$, we must have $f \in P^{\langle n+1\rangle}$. Given $\delta \in D^n$,

$$\delta(rf) = [\delta, r](f) + r\delta(f) \in P.$$

Since $rf \in P^{\langle n+1\rangle} \subseteq P^{\langle n\rangle}$, we have that $f \in P^{\langle n\rangle}$ by the induction hypothesis. Then, $[\delta, r](f) \in P$, because $[\delta, r] \in D^{n-1}$. We conclude that $r\delta(f) = \delta(rf) - [\delta, r](f) \in P$. Then, $r\delta(f) \in P$, and so, $\delta(f) \in P$, because P is a prime ideal and $r \notin P$. Hence, $f \in P^{\langle n+1\rangle}$. □

Proposition 1.35. Let R be a finitely generated k-algebra, I be an ideal of R, and n be a positive integer. Then $I^n \subseteq I^{\langle n\rangle}$.

Proof. Again, we use induction on n. The base case is straightforward: $I = I^{\langle 1\rangle}$ because $D^0 = R$.

Suppose that $I^n \subseteq I^{\langle n\rangle}$. Notice that I^n is generated by the elements of the form fg where $f \in I$, $g \in I^n$. In order to show that $I^{n+1} \subseteq I^{\langle n+1\rangle}$, it is enough to show that $fg \in I^{\langle n+1\rangle}$ for any such f and g.

To do that, we consider any $\delta \in D^n$, and we will show that $\delta(fg) \in I$. And in fact, since by induction hypothesis $g \in I^n \subseteq I^{\langle n\rangle}$, then

$$\delta(fg) = \underbrace{\underbrace{[\delta, f]}_{\in D^{n-1}}(\underbrace{g}_{\in I^{\langle n\rangle}})}_{\in I} + \underset{\in I}{f}\,\delta(g) \in I.$$

Notice here we used the fact that $\delta f = [\delta, f] + f\delta$. We conclude that $I^{n+1} \subseteq I^{\langle n+1\rangle}$. □

Lemma 1.36. For any radical ideal I and prime ideal P in a k-algebra R, $(I_P)^{\langle n\rangle} = \left(I^{\langle n\rangle}\right)_P$.

A lot more is true: taking differential powers commutes with localization at any multiplicative set W [BJNB19, Lemma 3.9].

Remark 1.37. Let k be a field, $R = k[x_1, \ldots, x_d]$ or $R = k[\![x_1, \ldots, x_d]\!]$, and $\mathfrak{m} = (x_1, \ldots, x_d)$. In this case,

$$D_R^n = R\left\langle \frac{1}{\alpha_1!}\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{1}{\alpha_d!}\frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} \;\middle|\; \alpha_1 + \ldots + \alpha_d \leqslant n \right\rangle.$$

If $f \notin \mathfrak{m}^n$, then f has a monomial of the form $x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ with nonzero coefficient $\lambda \in k$ for some $a_1 + \cdots + a_d < n$. Fix such a monomial which is minimal among all monomials appearing in f under the graded lexicographical order. Applying the differential operator $\frac{1}{\alpha_1!}\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{1}{\alpha_d!}\frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$ maps $\lambda x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ to the nonzero element $\lambda \in k$, and any other monomial appearing in f either to a nonconstant monomial or to zero. Consequently, $f \notin \mathfrak{m}^{\langle n \rangle}$. Hence, $\mathfrak{m}^{\langle n \rangle} \subseteq \mathfrak{m}^n$. Since $\mathfrak{m}^n \subseteq \mathfrak{m}^{\langle n \rangle}$ by Lemma 1.35, we conclude that $\mathfrak{m}^{\langle n \rangle} = \mathfrak{m}^n$.

There is a more technical version of this idea that proves that if we assume that k is perfect, and $\mathfrak{m}$ is any maximal ideal in R a regular algebra essentially of finite type over k, then we still have $\mathfrak{m}^n = \mathfrak{m}^{\langle n \rangle}$. This technical point, however, is difficult – this is the subtle part of the proof. For a complete proof, see [DSGJ, Theorem 3.6].

Theorem 1.38 (Zariski–Nagata Theorem for polynomial and power series rings [Zar49])**.** Let k be a perfect field, R be either $k[x_1, \dots, x_d]$ or $k[\![x_1, \dots, x_d]\!]$. For any prime ideal P and any maximal ideal $\mathfrak{m} \supseteq P$, we have $P^{(n)} \subseteq \mathfrak{m}^n$ for all $n \geqslant 1$. Moreover,

$$P^{(n)} = \bigcap_{\substack{\mathfrak{m} \supseteq P \\ \mathfrak{m} \in \operatorname{mSpec}(R)}} \mathfrak{m}^n.$$

Proof.

$$P^{(n)} \underset{1.34}{\subseteq} P^{\langle n \rangle} \underset{1.32}{\subseteq} \mathfrak{m}^{\langle n \rangle} \underset{1.37}{=} \mathfrak{m}^n.$$

For the converse, take $f \in \mathfrak{m}^n$ for all the maximal ideals $\mathfrak{m} \supseteq P$. For each maximal ideal $\mathfrak{m}$ containing P, we have $f \in \mathfrak{m}^{\langle n \rangle}$ by 1.37, so for every $\delta \in D^{n-1}$, $\delta(f) \in \mathfrak{m}$. But R/p is a finitely generated algebra over a field, and thus a Hilbert-Jacobson ring, which means that every prime ideal is an intersection of maximal ideals:

$$P = \bigcap_{\substack{\mathfrak{m} \supseteq P \\ \mathfrak{m} \in \operatorname{mSpec}(R)}} \mathfrak{m}.$$

But then we have $\partial(f) \in P$, so $f \in P^{\langle n \rangle}$. □

There are various extension of Zariski–Nagata. Eisenbud and Hochster showed that if P is a prime ideal in any noetherian ring R, we always have

$$P^{(n)} \supseteq \bigcap_{\substack{\mathfrak{m} \supseteq P \\ \mathfrak{m} \in \operatorname{mSpec}(R)}} \mathfrak{m}^n,$$

with equality if R is a regular ring [EH79]. Notice that this result does not require R to contain a field. As to the differential operators description of symbolic powers, there is also a mixed characteristic version, but it requires we consider more than differential operators. Over $\mathbb{Z}[x_1, \dots, x_d]$, for example, we also need some notion of differentiation by prime integers p, which can be done using Joyal and Buium's p-derivations [Joy85, Bui95], as shown in [DSGJ].

2 How do we actually compute symbolic powers?

In practice, the definition is not so useful to actually compute the symbolic powers of a given ideal, even over a polynomial ring. Taking the intersection of the powers of the corresponding maximal ideals can also be quite difficult, since unless we are talking about finite sets of points we want to intersect infinitely many ideals.

With a computer, we may find all the primary components of I and I^n and intersect the appropriate (and finitely many) components of I^n to obtain $I^{(n)}$, but determining the primary decomposition of an ideal is a notoriously difficult computational problem. In fact, finding a primary decomposition for a monomial ideal is an NP complete problem [HS02]. Some of the methods we will describe below are used by the `SymbolicPowers` package [DGSS17] for the commutative algebra software `Macaulay2` [GS].

Exercise 2.1. Use `Macaulay2` to find primary decompositions of I^2, I^3 and I^{10}, where I is each of the following ideals, and then use these decompositions to determine $I^{(2)}$, $I^{(3)}$ and $I^{(10)}$. Consider the fields $k = \mathbb{Q}, \mathbb{Z}/2$ and $\mathbb{Z}/101$.

(a) I the defining ideal of the curve (t^3, t^4, t^5) in $k[x, y, z]$.

(b) $I = (xy, yz, xz)$, in $k[x, y, z]$ and $k[x, y, z, u, v]$.

(c) $I = (x(y^3 - z^3), y(z^3 - x^3), z(x^3 - y^3))$ in $k[x, y, z]$.

(d) The ideal generated by all the degree 2 monomials in $k[x_1, \ldots, x_5]$.

Are there better methods you can use to determine the same symbolic powers using `Macaulay2`? If so, try asking Macaulay2 to compute the symbolic powers of the previous ideals using different methods. Did your answers change with the field?

There are however classes of ideals for which we can compute symbolic powers in ways that avoid determining a primary decomposition of I^n. We will now discuss some of them.

2.1 Monomial ideals

Example 2.2. Consider a field k and let $R = k[x, y, z]$. In Example 1.8 b, we found a primary decomposition for the following monomial radical ideal:

$$I = (xy, xz, yz) = (x, y) \cap (x, z) \cap (y, z).$$

When we localize at each of the associated primes of I, which are (x, y), (x, z) and (y, z), the third variable gets inverted, so that the remaining two components become the whole ring. Moreover, the pre-image of $(x, y)^n R_{(x,y)}$ in R is $(x, y)^n$. Thus the symbolic powers of I are given by

$$I^{(n)} = (x, y)^n \cap (x, z)^n \cap (y, z)^n.$$

In particular, $xyz \in I^{(2)}$. However, all homogeneous elements in I^2 have degree at least 4, since I is a homogeneous ideal generated in degree 2. Therefore, $xyz \notin I^2$, and $I^2 \neq I^{(2)}$. In fact, the maximal ideal (x, y, z) is an associated prime of I^2, since $(x, y, z) = (I^2 : xyz)$.

Exercise 2.3. If I is a squarefree monomial ideal in $k\,[x_1, \ldots, x_n]$, then I is a radical ideal whose minimal primes are generated by variables. Writing an irredundant decomposition $I = \bigcap_i Q_i$, where each Q_i is an ideal generated by variables, show that $I^{(n)} = \bigcap_i Q_i^n$.

For more on symbolic powers of monomial ideals, see [CEHH16].

2.2 Finite sets of points in $\mathbb{A}^n$ and $\mathbb{P}^n$

There are several examples of finite sets of points whose corresponding symbolic powers exhibit interesting behaviors.

Given a field k, an affine point P in $\mathbb{A}^n_k$ with coordinates $(a_1, \ldots, a_n)$ corresponds to the ideal $I(P) = (x_1 - a_1, \ldots, x_n - a_n)$ in $k[x_1, \ldots, x_n]$, and the point in projective space $\mathbb{P}^n_k$ with coordinates $(a_0 : \cdots : a_n)$ corresponds to the homogeneous ideal $(a_i x_0 - a_0 x_i, \ldots, a_i x_n - a_n x_i)$ in $k[x_0, \ldots, x_n]$ for any i such that $a_i \neq 0$. More generally, given a set of points $X = \{P_1, \ldots, P_p\}$ in either $\mathbb{A}^n_k$ or $\mathbb{P}^n_k$, the vanishing ideal of X is given by $I(X) = \cap_{i=1}^p I(P_i)$. In both the affine and projective settings, the symbolic powers of the set of points $I(X)$ are given by $I(X)^{(n)} = \cap_{i=1}^p I(P_i)^n$, the sets of polynomials that vanish up to order n in X.

2.3 Generic determinantal ideals

One of the few classes of ideals whose symbolic powers we can describe explicitly are generic determinantal ideals. In fact, there is an explicit description for the primary decomposition of all the powers of such ideals.

Example 2.4 (De Concini–Eisenbud–Procesi [DEP80])**.** Let k be a field of characteristic 0 or $p > \min\{t, n-t, m-t\}$. Consider a generic $n \times m$ matrix X, with $n \leqslant m$, the polynomial ring $R = k[X]$ generated by all the variables in X, and the ideal $I = I_t(X)$ generated by the $t \times t$ minors of X, where $2 \leqslant t \leqslant n$.

The products of the form $\Delta = \delta_1 \cdots \delta_k$, where each δ_i is an s_i-minors of X, generate $R = k[X]$ as a vector space over k. More importantly, an interesting subset of such products, known as standard monomials, form a k-basis for R. These are enough to both describe the symbolic powers of $I = I_t(X)$ and to give explicit primary decompositions for all powers of I. Given a product $\Delta = \delta_1 \cdots \delta_k$ as above, $\Delta \in I^{(r)}$ if and only if

$$\sum_{i=1}^{k} \max\{0, s_i - t + 1\} \geqslant r.$$

Note also that all $s_i \leqslant n$, since there are no minors of larger sizes. Moreover, $I^{(r)}$ is generated by all the $\Delta \in I^{(r)}$ of this form. In particular, note that multiplying such a Δ by minors of size $\leqslant t-1$ does not affect whether or not $\Delta \in I^{(n)}$. Moreover, I^s has the following primary decomposition:

$$I^s = \bigcap_{j=1}^{t} (I_j(X))^{((t-j+1)s)} = (I_1(X))^{(ts)} \cap \cdots \cap (I_{t-1}(X))^{(2s)} \cap (I_t(X))^{(s)}.$$

To obtain an irredundant primary decomposition, we take the previous decomposition and drop the terms in $I_j(X)$ for $j < n - s(n-t)$.

There are similar formulas for when for the ideal of $t \times t$ minors of a symmetric $n \times n$ matrix [JMnV15, Proposition 4.3 and Theorem 4.4] or the ideal of $2t$-Pfaffians of a generic $n \times n$ matrix [DN96, Theorem 2.1 and Theorem 2.4]. For an in-depth treatment of determinantal ideals, see [BV88].

Exercise 2.5. Let $I = I_2(X)$, where X is a generic 3×3 matrix. Find generators for $I^{(2)}$.

Exercise 2.6. Show that if I is the ideal in $k[X]$ generated by the maximal minors of a generic matrix X, where k satisfies the conditions of Example 2.4, then $I^n = I^{(n)}$ for all $n \geqslant 1$.

Note, however, that this does not give any information about the symbolic powers of ideals generated by the minors of a matrix outside of the generic case.

Example 2.7. In Example 1.25, we saw that $P^{(2)} \neq P^2$ for the prime ideal

$$P = \left(x^2y - z^2, xz - y^2, yz - x^3\right)$$

in $R = k[x, y, z]$. This is the ideal generated by the 2×2 of the matrix

$$\begin{pmatrix} x^2 & y & z \\ z & x & y \end{pmatrix}.$$

In fact, as we will see in Theorem 3.2, $P^{(n)} \neq P^n$ for all n. In contrast, by Exercise **??** symbolic and ordinary powers of ideals of maximal minors of generic matrices always coincide.

2.4 Saturations

In general, symbolic powers are always given by saturations.

Definition 2.8. Let I, J be ideals in a noetherian ring R. The *saturation* of I with respect to J is the ideal given by

$$(I : J^\infty) := \bigcup_{n \geqslant 1} (I : J^n) = \{r \in R : rJ^n \subseteq I \text{ for some } n \geqslant 1\}.$$

Remark 2.9. Note that the colon ideals $(I : J^n)$ form an increasing chain that must then stabilize, so that $(I : J^\infty) = (I : J^n)$ for some n. Computationally, this can be computed very easily by taking the successive colons $(I : J^n)$ until they stabilize.

Lemma 2.10. Let I be an ideal in a noetherian ring R with no embedded primes. There exists an ideal J such that for all $n \geqslant 1$,

$$I^{(n)} = (I^n : J^\infty).$$

This ideal J can be taken to be:

(a) The principal ideal $J = (s)$ generated by an element $s \in R$ that is not contained in any minimal prime of I, but that is contained in all the embedded primes of I^n for all $n \geqslant 1$.

(b) the intersection of all the non-minimal primes in $A(I)$;

(c) the intersection of any finite set of primes $P \supseteq I$ that are not minimal over I, as long as this set includes all of the non-minimal primes in $A(I)$.

Proof. Recall that we use the notation $A(I)$ to refer to set of primes that are associated to some power of I (cf. 1.11), which is a finite set by Theorem 1.12. If $A(I)$ consists only of minimal primes of I, then all symbolic and ordinary powers of I coincide, so that we can take $J = R$. Otherwise, let $P_1, \ldots, P_k$ be the primes in $A(I)$ that are not minimal over I. Consider an element s not in any minimal prime of I and such that $s \in P_1 \cap \cdots \cap P_k$. For each n, write

$$I^n = I^{(n)} \cap Q_1 \cap \cdots \cap Q_t,$$

where each Q_j is a primary ideal with radical one of the P_i. Then, since $s \in \sqrt{Q_i}$ for all i, we have $(Q_i : s^\infty) = R$, and thus

$$(I^n : s^\infty) = \left(I^{(n)} : s^\infty\right) \cap (Q_1 : s^\infty) \cap \cdots \cap (Q_t : s^\infty) = (I^{(n)} : s^\infty).$$

Moreover, $I^{(n)}$ is an intersection of primary ideals, the radicals of which do not contain s. Therefore, $(I^{(n)} : s^\infty) = I^{(n)}$. This shows part a).

Now take J to be the intersection of all the non-minimal primes in $A(I)$. Since $s \in J$, then

$$(I^n : J^\infty) \subseteq (I^n : s^\infty) = I^{(n)}.$$

Moreover, writing an irredundant primary decomposition

$$I^n = I^{(n)} \cap Q_1 \cap \cdots \cap Q_t,$$

we have $J \subseteq \sqrt{Q_1} \cap \cdots \cap \sqrt{Q_t}$. Therefore, there exists a power of J, say J^k, that is contained in $Q_1 \cap \cdots \cap Q_t$, so that $I^{(n)} J^k \subseteq I^n$. This proves part b):

$$I^{(n)} \subseteq (I^n : J^k) \subseteq (I^n : J^\infty).$$

Finally, we may also take J to be the intersection of any finite set of primes $P \supseteq I$ that are not minimal over I, as long as this set includes all of the non-minimal primes in $A(I)$. Indeed, above we only used two facts: that J contains some element not in any minimal prime of I, and that $J \subseteq \sqrt{Q}$ whenever Q is a non-minimal but irredundant primary component of I^n for some n. $\square$

Exercise 2.11. Let $(R, \mathfrak{m})$ be a local ring and P a prime ideal of height $\dim R - 1$. Show that $P^{(n)} = (P^n : \mathfrak{m}^\infty)$ for all $n \geqslant 1$.

Unfortunately, finding J as in Lemma 2.10 requires in principle some concrete knowledge of $A(I)$. Knowing an upper bound for the value n at which $\operatorname{Ass}(R/I^n)$ stabilizes would suffice, but there are essentially no effective bounds to find such a value. Moreover, the number of associated primes of a power of a prime ideal can be arbitrarily large [KS19].

Definition 2.12 (Jacobian ideal)**.** Let k be a field and $R = k[x_1, \ldots, x_n]/I$, where $I = (f_1, \ldots, f_r)$ has pure height h. The *Jacobian matrix* of R is the matrix given by

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ & \ddots & \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_n} \end{pmatrix}.$$

The *jacobian ideal* of R is the ideal generated by the h-minors of the Jacobian matrix.

Turns out the Jacobian ideal is indeed well-defined – meaning, our definition does not depend on the choice of presentation for R – and that it determines the smooth locus of R. For proofs of these well-known facts, see [Eis95, Section 16.6].

Theorem 2.13 (Jacobian criterion)**.** Let $R = k[x_1, \ldots, x_n]/I$ with k a perfect field, and assume that I has pure height h. The Jacobian ideal J defines the smooth locus of R: a prime P contains J if and only if R_P is not a regular ring.

Proof. See [Eis95, Corollary 16.20]. □

This gives us access to a way to compute symbolic powers of ideals of pure height over a polynomial ring.

Lemma 2.14. Let $R = k[x_1, \ldots, x_n]$ with k a perfect field, and let $I = (f_1, \ldots, f_r)$ be an ideal of pure height h. Let

$$J = I_h \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ & \ddots & \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_n} \end{pmatrix}$$

be the ideal generated by the h-minors of the generators of I. If $t \in J$ is not in any minimal prime of I, then for all $n \geqslant 1$,

$$I^{(n)} = (I^n : t^\infty) \, .$$

Proof. By Lemma 2.10, we only need to show that such a t is contained in every prime P that is embedded to some I^n. Suppose that P is indeed an embedded prime of I^n for some n. Then P_P is an embedded prime of I_P^n, which implies that $I_P^n \neq I_P^{(n)}$. In particular, I_P is not a complete intersection, by Theorem 1.22, and thus $(R/I)_P$ cannot be regular ring. By Theorem 2.13, $P \supseteq J \ni t$. □

3 Open Problems

3.1 The (in)equality of symbolic and ordinary powers

In general, the question of when the symbolic and ordinary powers of a given ideal coincide is open. There are conditions on I that are equivalent to $I^{(n)} = I^n$ for all $n \geqslant 1$ given by Hochster [Hoc73] when I is prime, and extended by Li and Swanson [LS06] to the case when I is a radical ideal. However, even thought their conditions hold over any noetherian ring, their conditions are not easy to check in practice, nor to describe here.

Question 3.1. Let R be a regular ring. For which ideals I with no embedded primes in R do we have $I^{(n)} = I^n$ for all $n \geqslant 1$? Is there an invariant d depending on the ring R or the ideal I such that $I^{(n)} = I^n$ for all $n \leqslant d$ (or for $n = d$) implies that $I^{(n)} = I^n$ for all $n \geqslant 1$?

There are some settings under which this is understood. The following is [Hun86, Corollary 2.5]:

Theorem 3.2 (Huneke, 1986)**.** Let R be a regular local ring of dimension 3, and P a prime ideal in R of height 2. The following are equivalent:

(a) $P^{(n)} = P^n$ for all $n \geqslant 1$;

(b) $P^{(n)} = P^n$ for some $n \geqslant 2$;

(c) P is generated by a regular sequence.

In particular, for a height 2 prime P in a regular local ring of dimension 3, we have $P^{(n)} \neq P^n$ for all $n \geqslant 2$ as long as P has at least 3 generators. In dimension higher than 3, we can find prime ideals that are not generated by regular sequences whose symbolic and ordinary coincide nevertheless.

Example 3.3 (Example 4.4 in [HH92], special case of [Sch91] and Corollary 4.3 in [GH19])**.** Consider the prime P in $R = k[x, y, z, w]$ that is given as the kernel of the map $R \longrightarrow k[s, t]$ that takes $x \mapsto s^3$, $y \mapsto s^2t$, $z \mapsto st^2$ and $w \mapsto t^3$, where k is any field. Then $P^{(n)} = P^n$ for all $n \geqslant 1$. However, P is a prime of height 2 that is minimally generated by 3 elements, and thus not generated by a regular sequence.

Theorem 3.4 (see Theorem 2.3 in [CFG⁺16], also [Mor99, HU89])**.** Let $R = k[x_0, \ldots, x_n]$ be a polynomial ring over a field k. Let I be a height 2 ideal in R such that R/I is Cohen-Macaulay and such that I_P is generated by a regular sequence for all primes $P \neq (x_0, \ldots, x_n)$ containing I. Then $I^{(k)} = I^k$ for all $k < n$ regardless of the minimal number of generators of I. Moreover, the following statements are equivalent:

(a) $I^{(k)} = I^k$ for all $k \geqslant 1$;

(b) $I^{(n)} = I^n$;

(c) I is generated by at most n elements.

Remark 3.5. Notice that if P is a height 2 prime ideal in a polynomial ring in 3 variables, meaning that $n = 2$ in the statement of Theorem 3.4, then the conclusions of Theorems 3.2 and 3.4 coincide, although Theorem 3.2 also adds the equivalence with condition

(d) $I^{(k)} = I^k$ for some $k \geqslant 2$;

This suggests that Theorem 3.4 might hold if we add condition (d) to the equivalences stated.

The problem of equality of symbolic and ordinary powers of ideals is also understood more generally for licci prime ideals [HU89, Corollary 2.9]. For primes of height $\dim R - 1$, equality of *all* symbolic and ordinary powers is equivalent to the ideal being a complete intersection.

Theorem 3.6 (Cowsik–Nori [CN76])**.** Let R be a Cohen-Macaulay local ring and let P be a prime ideal such that R_P is a regular ring. If R/P^n is Cohen-Macaulay for all $n \geqslant 1$, then P is generated by a regular sequence.

Exercise 3.7. Let R be a regular local ring and P be a prime ideal such that $\dim(R/P) = 1$. Show that $P^{(n)} = P^n$ for all $n \geqslant 1$ if and only if P is generated by a regular sequence.

Characterizing which squarefree monomial ideals have $I^{(n)} = I^n$ for all $n \geqslant 1$ is still an open question. However, it is conjectured that this condition is equivalent to I being packed.

Definition 3.8 (König ideal)**.** Let I be a squarefree monomial ideal of height c in a polynomial ring over a field. We say that I könig if I contains a regular sequence of monomials of length c.

Despite the fact that all squarefree monomial ideals do contain a regular sequence of length equal to their height, not all squarefree monomial ideals are könig.

Exercise 3.9. Show that (xy, xz, yz) is not könig.

Definition 3.10 (Packed ideal)**.** A squarefree monomial ideal of height c is said to be packed if every ideal obtained from I by setting any number of variables equal to 0 or 1 is könig.

Exercise 3.11. Give an example of an ideal that is packed and of one that is not packed.

The following is a restatement by Gitler, Valencia, and Villarreal in the setting of symbolic powers of a conjecture of Conforti and Cornuéjols about max-cut min-flow properties.

Conjecture 3.12 (Packing Problem)**.** Let I be a squarefree monomial ideal in a polynomial ring over a field k. The symbolic and ordinary powers of I coincide if and only if I is packed.

The difficult direction is to show that if I is packed, then $I^{(n)} = I^n$ for all $n \geqslant 1$.

Exercise 3.13. Let I be a squarefree monomial ideal. Show that if $I^{(n)} = I^n$ for all $n \geqslant 1$ then I must be packed.

The Packing Problem has been solved for the case when I is the edge ideal of a graph [GVV07].

Definition 3.14 (Edge ideal)**.** Let G be a simple graph on n vertices $\{v_1, \ldots, v_n\}$. Given a field k, the *edge ideal* of G in $k[x_1, \ldots, x_n]$ is the ideal

$$I = \left(x_i x_j \,\middle|\, \text{if there is an edge between the vertices } v_i \text{ and } v_j\right).$$

Theorem 3.15 (Gitler–Valencia–Villareal, [GVV07])**.** Let I be the edge ideal of a graph G. The following are equivalent:

(a) G is a bipartite graph;

(b) $I^{(n)} = I^n$ for all $n \geqslant 1$;

(c) I is packed.

While the more general version of the Packing Problem is still open, the question of whether it is sufficient to test $I^{(n)} = I^n$ only for finitely many values of n is settled for monomial ideals.

Theorem 3.16 (Núñez Betancourt – Montaño [MnNnB19])**.** Let I be a squarefree monomial ideal generated by μ elements. If $I^{(n)} = I^n$ for all $n \leqslant \frac{\mu}{2}$, then $I^{(n)} = I^n$ for all $n \geqslant 1$.

3.2 What is the degree of an element in $I^{(n)}$?

When I is a homogeneous ideal in a graded ring, the symbolic powers of I are also homogeneous ideals. It is then natural to ask what is the minimal degree of an element in $I^{(n)}$ for each n. If I corresponds to a finite set of points in $\mathbb{P}^N$, this amounts to asking what is the smallest degree of a hypersurface passing through each of the given points with multiplicity n.

Given a homogeneous ideal in $R = k[x_0, \ldots, x_N]$, we write $\alpha(I)$ to denote the minimal degree of an element in I.

Conjecture 3.17 (Nagata [Nag65])**.** If I defines $n \geqslant 10$ very general points in $\mathbb{P}^2_{\mathbb{C}}$,

$$\alpha(I^{(m)}) > m\sqrt{n}.$$

This question remains open except for some special cases.

Conjecture 3.18 (Chudnovsky)**.** Let X be a finite set of points in $\mathbb{P}^N$, and $I = I(X)$ be the corresponding ideal in $k[x_0, \ldots, x_N]$. Then

$$\frac{\alpha(I^{(m)})}{m} \geqslant \frac{\alpha(I) + N - 1}{N}.$$

Turns out that the limit of the right hand side exists and equals the infimimum on the same set. More precisely,

$$\hat{\alpha}(I) = \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m} = \inf_m \frac{\alpha(I^{(m)})}{m}.$$

We can restate Chudnovsky's conjecture in terms of this constant $\hat{\alpha}$, known as the Waldschmidt constant of I. More precisely, Chudnovsky's conjecture asks if

$$\hat{\alpha}(I^{(m)}) \geqslant \frac{\alpha(I)+N-1}{N}.$$

This conjecture has been shown for finite sets of very general points in $\mathbb{P}_k^N$ as long as k is an algebraically closed field [FMX18, Theorem 2.8]. One might wonder if we can extend this to any homogeneous ideal, perhaps by substituting N by the height*The most appropriate invariant is likely to be one we will define later, known as the* big height*.* of I, a fact which has been shown to hold for squarefree monomial ideals [BCG+16, Theorem 5.3]. Chudnovsky's Conjecture is essentially open otherwise. Chudnovsky's Conjecture is a special case of a more general conjecture by Demailly [Dem82].

3.3 The Eisenbud–Mazur Conjecture

While $I^{(2)} \subseteq I$ always holds, we may wonder whether $I^{(2)}$ may contain a minimal generator of I.

Conjecture 3.19 (Eisenbud–Mazur [EM97])**.** Let $(R, \mathfrak{m})$ be a localization of a polynomial ring over a field k of characteristic 0. If I is a radical ideal in R, then $I^{(2)} \subseteq \mathfrak{m}I$.

This fails in prime characteristic:

Example 3.20 (Eisenbud–Mazur [EM97])**.** Let p be a prime integer, and let I be the kernel of the map

$$\begin{aligned} \mathbb{F}_p[x_1,x_2,x_3,x_4] &\longrightarrow \mathbb{F}_p[t] \\ x_1 &\longmapsto t^{p^2} \\ x_2 &\longmapsto t^{p(p+1)} \\ x_3 &\longmapsto t^{p^2+p+1} \\ x_4 &\longmapsto t^{(p+1)^2}. \end{aligned}$$

Consider the polynomial $f = x_1^{p+1}x_2 - x_2^{p+1} - x_1x_3^p + x_4^p \in I$. Note that f is a quasi-homogeneous polynomial, and in fact $f \in I^{(2)}$. To see that, consider

$$\begin{aligned} g_1 &= x_1^{p+1} - x_2^p \in I, \\ g_2 &= x_1x_4 - x_2x_3 \in I, \\ g_3 &= x_1^p x_2 - x_3^p \in I, \end{aligned}$$

and note that

$$x_1^p f = g_1g_3 + g_2^p \in I^2.$$

We also claim that f is in fact a minimal generator of I, meaning $f \notin (x_1,x_2,x_3,x_4)I$. To do this, one can show that no element of I contains a term of the form x_4^a for any $1 \leqslant a < p$, and since I is generated by binomials, it suffices to show there is no element of the form $x_4^a - x_3^b x_2^c x_1^d$ in I. We leave this as an exercise.

The Eisenbud–Mazur Conjecture also fails if the ring is not regular. It is still open in most cases over fields of characteristic 0.

Exercise 3.21. Show the Eisenbud–Mazur conjecture for squarefree monomial ideals.

More generally, Eisenbud and Mazur showed that if I in a monomial ideal and P is a monomial prime containing I, then $I^{(d)} \subseteq PI^{(d-1)}$ for all $d \geqslant 1$ [EM97, Proposition 7]. They also show Conjecture 3.19 for licci ideals [EM97, Theorem 8] and quasi-homogeneous unmixed ideals in equicharacteristic 0 [EM97, Theorem 9]. For more on the status of this conjecture, see [DDSG$^+$18, Section 2.3].

3.4 Symbolic Rees algebras

The symbolic powers of an ideal I form a graded family,[1] which allows us to package them together in a single graded object, called the symbolic Rees algebra (or symbolic blowup) of I.

Definition 3.22 (Symbolic Rees algebra)**.** Let R be a ring and I an ideal in R. The *symbolic Rees algebra* of I is the graded algebra

$$\mathcal{R}_s(I) := \bigoplus_{n \geqslant 0} I^{(n)} t^n \subseteq R[t],$$

where the indeterminate t is used to keep track of the grading.

This mimics the construction of the usual Rees algebra of I, $\bigoplus_n I^n t^n$. But unlike the Rees algebra, the symbolic Rees algebra may fail to be a noetherian ring.

Exercise 3.23. Show that the symbolic Rees algebra of an ideal I in a ring R is a finitely generated R-algebra if and only if it is a noetherian ring.

Exercise 3.24. If the symbolic Rees algebra of an ideal I in a ring R is finitely generated, show that there exists k such that $I^{(kn)} = \left(I^{(k)}\right)^n$ for all $n \geqslant 1$. The converse also holds as long as R is excellent.

Which ideals do have a noetherian symbolic Rees algebra? For example, the symbolic Rees algebra of a monomial ideal is noetherian [Lyu88, Proposition 1]. What is maybe more surprising is that symbolic Rees algebras are often not finitely generated. The first example of this is due to Rees [Ree58], and Roberts showed this can happen even when R is a regular ring [Rob85], building on Nagata's counterexample to Hilbert's 14th Problem [Nag65]. Roberts' example gave a negative answer to the following question of Cowsik:

Question 3.25 (Cowsik)**.** Let P be a prime ideal in a regular ring R. Is the symbolic Rees algebra of P always a noetherian ring, or equivalently, a finitely generated R-algebra?

[1]That is,$I^{(a)} I^{(b)} \subseteq I^{(a+b)}$ for all a and b.

Cowsik's motivation was a result of his [Cow84] showing that a positive answer would imply that all such primes are set-theoretic complete intersections, that is, complete intersections up to radical. Eliahou, Huckaba, Huneke, Vasconcelos and others proved various criteria that imply noetherianity. Space monomial curves (t^a, t^b, t^c), however, were known to be set-theoretic complete intersections [Bre79, Her80, Val81], and much work was devoted to studying their symbolic Rees algebras. Surprisingly, the answer to Cowsik's question is negative even for this class of primes, with the first non-noetherian example found by Morimoto and Goto [GM92]. In [Cut91], Cutkosky gives criteria for the symbolic Rees algebra of a space monomial curve to be noetherian, and in particular shows that the symbolic Rees algebra of $k[t^a, t^b, t^c]$ is noetherian when $(a + b + c)^2 > abc$. There is a vast body of literature on the case of ideals defining space monomial curves (t^a, t^b, t^c) alone [Cut91, Mor91, GNS91b, GM92, GNW94, GNS91a, HU90, Sri91].

4 The Containment Problem

The Containment Problem for I is an attempt to compare the symbolic and ordinary powers of I. The past two decades have seen a rush of activity around this question.

4.1 A famous containment

The containment $I^a \subseteq I^{(b)}$ holds if and only if $a \geqslant b$. Containments of type $I^{(a)} \subseteq I^b$ are a lot more interesting.

Question 4.1 (Containment Problem)**.** Let R be a noetherian ring and I be an ideal in R. When is $I^{(a)} \subseteq I^b$?

This packages together a few different questions. First, this contains the equality problem, since the b-th symbolic and ordinary powers coincide if and only if $I^{(b)} \subseteq I^b$. When the answer is no, the containment problem is one way to measure how far the symbolic and ordinary powers are from each other. If we do have a particular answer to the containment problem, say $I^{(a)} \subseteq I^b$, this gives us lower bounds for the degrees of the elements in $I^{(a)}$. Indeed, this containment implies

$$\alpha\left(I^{(a)}\right) \geqslant \alpha\left(I^b\right) = b\alpha(I).$$

In general, the Containment Problem can be quite difficult, although we can answer it completely if we have an explicit description of both the symbolic and ordinary powers of our ideal. Having an explicit description of symbolic powers, however, is fairly rare.

Exercise 4.2. Solve the containment problem for generic determinantal ideals.

Exercise 4.3. The second symbolic power of the monomial ideal $I = (xy, xz, yz) \subseteq k[x, y, z]$ does not coincide with its square. However, show that the containment $I^{(3)} \subseteq I^2$ does hold.

Over a Gorenstein ring, the containment problem can be rephrased as a homological question, a fact first applied by Alexandra Seceleanu in [Sec15] and later used in [Gri18, Chapter 3] and [Gri] to study the containment problem for ideals generated by 2×2 minors of 2×3 matrices in dimension 3.

Exercise 4.4. Let $(R, \mathfrak{m})$ be a Gorenstein local ring and P a prime ideal of height $\dim R - 1$. Given $a \geqslant b$, show that $P^{(a)} \subseteq P^b$ if and only if the map $\operatorname{Ext}^d_R(R/P^b, R) \to \operatorname{Ext}^d_R(R/P^a, R)$ induced by the canonical projection vanishes.

But does Question 4.1 always make sense? That is, given b, must there exist an a such that $I^{(a)} \subseteq I^b$? If so, then the two graded families of ideals $\{I^n\}$ and $\left\{I^{(n)}\right\}$ are cofinal, and thus induce equivalent topologies. In 1985, Schenzel [Sch85] gave a characterization of when $\{I^n\}$ and $\left\{I^{(n)}\right\}$ are cofinal. In particular, if R is a regular ring and I is a radical ideal in R, then $\{I^n\}$ and $\left\{I^{(n)}\right\}$ are cofinal. Schenzel's characterization did not, however, provide information on the relationship between a and b.

It was not until the late 90s that Irena Swanson showed that the I-adic and I-symbolic topologies are equivalent if and only if they are linearly equivalent.[2]

[2]Word of caution: the words *linearly equivalent* have been used in the past to refer to other condition. For example, Schenzel used this term to refer to $I^{(n+k)} \subseteq I^n$ for all $n \geqslant 1$ and some constant k.

Theorem 4.5 (Swanson, 2000, [Swa00])**.** Let R be a noetherian ring, and I and J two ideals in R. The following are equivalent:

(i) $\{I^n : J^\infty\}$ is cofinal with $\{I^n\}$.

(ii) There exists an integer c such that $(I^{cn} : J^\infty) \subseteq I^n$ for all $n \geqslant 1$.

In particular, given a radical ideal in a regular ring, there exists an integer c such that $I^{(cn)} \subseteq I^n$ for all $n \geqslant 1$. More surprisingly, over a regular ring this constant can be taken uniformly, meaning depending only on R.

Definition 4.6 (Big height)**.** Let I be an ideal with no embedded primes. The *big height*[3] of I is the maximal height of an associated prime of I. If the big height of I coincides with the height of I, meaning that all associated primes of I have the same height, we say that I has *pure height*.

Theorem 4.7 (Ein–Lazarsfeld–Smith, Hochster–Huneke, Ma–Schwede [ELS01, HH02, MS18a])**.** Let R be a regular ring and I a radical ideal in R. If h is the big height of I, then $I^{(hn)} \subseteq I^n$ for all $n \geqslant 1$.

Remark 4.8. Equivalently, $I^{(n)} \subseteq I^{\lfloor \frac{n}{h} \rfloor}$ for all $n \geqslant 1$.

We cannot replace big height by height in Theorem 4.7.

Example 4.9. Consider the ideal

$$I = (x,y) \cap (y,z) \cap (x,z) \cap (a) = (xya, xza, yza) \subseteq k[x,y,z,a],$$

which has height 1 and big height 2. If we replaced big height by height in Theorem 4.7, we would have $I^{(n)} = I^n$ for all $n \geqslant 1$. However, similarly to Example 2.2, $I^{(2)} \neq I^2$. Indeed, note that

$$xyza^2 \in I^{(2)} = (x,y)^2 \cap (y,z)^2 \cap (x,z)^2 \cap (a)^2,$$

whereas all elements in I^2 must have degree at least 6.

Exercise 4.10. Given integers $c < h$, construct an ideal I with height c and big height h in a polynomial ring such that $I^{(cn)} \nsubseteq I^n$ for some n. Hint: you may want to use Exercise 4.51.

Ein, Lazarsfeld, and Smith first proved Theorem 4.7 in the equicharacteristic 0 geometric case, using multiplier ideals. Hochster and Huneke then used reduction to characteristic p and tight closure techniques to prove the result in the equicharacteristic case. Recently, Ma and Schwede built on ideas used in the recent proof of the Direct Summand Conjecture to define a mixed characteristic analogue of multiplier/test ideals, allowing them to deduce the mixed characteristic version of Theorem 4.7.

Given an ideal I and $t \geqslant 0$, the multiplier ideal $\mathcal{J}(R, I^t)$ measures the singularities of $V(I) \subseteq \operatorname{Spec}(R)$, scaled by t in a certain sense. We refer to [ELS01, MS18a] for the definition. The proof of Theorem 4.7 in the characteristic 0 case relies on a few key properties of multiplier ideals:

[3] According to google, 6'2".

- $I \subseteq \mathcal{J}(R, I)$;
- For all $n \geqslant 1$, $\mathcal{J}\left(R, \left(P^{(nh)}\right)^{\frac{1}{n}}\right) \subseteq P$ whenever P is a prime of height h;
- For all integers $n \geqslant 1$, $\mathcal{J}\left(R, I^{tn}\right) \subseteq \mathcal{J}\left(R, I^{t}\right)^{n}$.

Then, given a prime ideal P of height h,

$$P^{(hn)} \subseteq \mathcal{J}\left(R, \left(P^{(nh)}\right)\right) \subseteq \mathcal{J}\left(R, \left(P^{(nh)}\right)^{\frac{1}{n}}\right)^{n} \subseteq P^{n}.$$

In characteristic p, a similar proof works, replacing multiplier ideals by test ideals.

Remark 4.11. As a corollary of Theorem 4.7, we obtain a uniform constant c as in Theorem 4.5. Indeed, the big height of any ideal is at most the dimension d of the ring, so that $I^{(dn)} \subseteq I^{n}$ for all n. This constant can be improved to $d-1$, since the ordinary and symbolic powers of any maximal ideal coincide.

In the non-regular setting, we do know that the two topologies are equivalent for all prime ideals in a fairly general setting:

Theorem 4.12 (Huneke–Katz–Validashti, Proposition 2.4 in [HKV09])**.** Let R be a complete local domain. For all prime ideals P, there exists a constant h such that $P^{(hn)} \subseteq P^{n}$.

However, the question of whether this constant h can be taken independently of P is still open in the non-regular setting. Since this asks for a uniform comparison between the symbolic and adic topologies, rings with this property are said to satisfy the Uniform Symbolic Topologies Property.

Question 4.13 (Uniform Symbolic Topologies)**.** Let R be a complete local domain. Is there a uniform constant h depending only on R such that

$$P^{(hn)} \subseteq P^{n}$$

for all primes P and all $n \geqslant 1$?

The answer is known to be yes in some special settings.

Theorem 4.14 (Huneke–Katz–Validashti, 2009, [HKV09])**.** Let R be an equicharacteristic reduced local ring such that R is an isolated singularity. Assume either that R is equidimensional and essentially of finite type over a field of prime characteristic zero, or that R has positive characteristic and is F-finite. Then there exists $h \geqslant 1$ with the following property: for all ideals I with positive grade for which the I-symbolic and I-adic topologies are equivalent, $I^{(hn)} \subseteq I^{n}$ holds for all $n \geqslant 1$.

Their result does not provide effective bounds for what h might be. In general, finding explicit best possible bounds for this constant is a very difficult question. For the case of monomial prime ideals over normal toric rings, see the work of Robert M. Walker [Wal16, Wal18b].

Example 4.15 (Carvajal-Rojas — Smolkin, 2018)**.** Let k be a field of characteristic p and consider $R = k[a, b, c, d]/(ad - bc)$. Then for all primes P in R, $P^{(2n)} \subseteq P^{n}$ for all $n \geqslant 1$.

4.2 Characteristic p is your friend

There are characteristic free questions that are easier to attack using positive characteristic techniques. Hochster and Huneke's proof that $I^{(hn)} \subseteq I^n$ for regular rings is an example of this [HH02]. More surprisingly, a characteristic p solution to a question can sometimes be enough to solve the equicharacteristic 0 case, via a method known as reduction to positive characteristic. We will now focus on the containment problem for radical ideals over a regular ring of characteristic p, and go over Hochster and Huneke's proof of Theorem 4.7.

When dealing with rings of prime characteristic p, we gain a powerful tool:

Definition 4.16. Let R be a ring of prime characteristic p. The *Frobenius* map is the R-homomorphism defined by $F(x) = x^p$. We denote the e-th iteration of the Frobenius map, $F^e(x) = x^{p^e}$, by F^e. Applying the e-iteration of Frobenius to an ideal I in R returns an ideal, the e-th *Frobenius power* of I, which we denote by $I^{[p^e]}$:

$$I^{[p^e]} := \left(a^{p^e} : a \in I\right).$$

We write $F^e_*(R)$ to denote the natural image of F^e. This is an R-module with underlying abelian group R, but R-module structure determined by the action of F: $r \cdot F^e_*(s) = F^e_*(r^{p^e} s)$.

Remark 4.17. If $I = (a_1, \ldots, a_n)$, then $I^{[p^e]} = \left(a_1^{p^e}, \ldots, a_n^{p^e}\right)$.

In prime characteristic commutative algebra, one often studies ring-theoretic properties of R by studying the module structure of $F^e_*(R)$. Many interesting singularities can be identified via the R-module structure of $F^e_*(R)$; these are known as F-singularities.

We will be focusing on regular rings of prime characteristic. One of the main facts we will need is that over a regular ring, the Frobenius map is flat. This is also one of the points where the assumption that we are working over a regular ring is crucial: the flatness of Frobenius *characterizes* regular rings.

Theorem 4.18 (Kunz, 1969 [Kun69])**.** If R is a reduced local ring of prime characteristic p. The following are equivalent:

- R is a regular ring.
- R is flat over R^p.
- $F^e_*(R)$ is a flat R-module.

This theorem has many important consequences.

Lemma 4.19. Let R be a regular ring of characteristic p. For all ideals I and J in R and all $q = p^e$,

$$(J : I)^{[q]} = \left(J^{[q]} : I^{[q]}\right).$$

Proof. Since $F^e_*(R)$ is a flat R-module by Theorem 4.18, this is a particular case of [Mat89, Theorem 7.4 (iii)]. ☐

Lemma 4.20. If R is a regular ring of characteristic p, the Frobenius map preserves associated primes, that is, $\operatorname{Ass}(R/I) = \operatorname{Ass}\left(R/I^{[q]}\right)$ for all $q = p^e$.

Proof. The Frobenius map is exact [Kun69, Theorem 2.1], and thus it takes minimal free resolutions to minimal free resolutions. In particular, if Q is a prime ideal in R, the Frobenius map takes a minimal free resolution of $(R/I)_Q$ to a minimal free resolution of $\left(R/I^{[q]}\right)_Q$. Moreover, the length of the resolution is preserved, so that the projective dimensions coincide. By the Auslander-Buchsbaum formula, this implies that the depths also coincide, so that

$$\operatorname{depth}(R/I)_Q = 0 \text{ if and only if } \operatorname{depth}\left(R/I^{[q]}\right)_Q = 0.$$

By Lemma 1.10, this completes the proof. □

Remark 4.21. One of the key ingredients we will need to show $I^{(hn)} \subseteq I^n$ is understanding the minimal number of generators of I after localizing at each associated prime. If $I = Q$ is a prime ideal of height h, then the only associated prime of Q is Q itself, and Q_Q is the maximal ideal of a regular local ring of dimension h, so that it is minimally generated by h elements. For a radical ideal I of big height h, I_P is generated by at most h elements when localized at any of its associated primes P. Indeed, since I is radical, $I = P \cap J$, where J contains elements not in P, and thus $I_P = P_P$, which is generated by as many elements as the height of P. By definition, this is at most h.

The results in [HH02] cover a more general case: there is no assumption that I is radical. The main ideas are still the same, but the maximal value for the minimal number of generators of I_P, when P runs through the associated primes of I, is no longer necessarily h. To overcome this issue, we can substitute I_P by a minimal reduction of I_P, which is generated by as many elements as the analytic spread of I_P. We will not discuss this in detail here, but this number is at most the height of P. The general form of Theorem 4.7 is then as follows: $I^{(hn)} \subseteq I^n$ for all $n \geqslant 1$, where h can be taken to be (by increasing order of refinement)

- the maximum value of the minimal number of generators of I_P, where P runs through the associated primes of I,
- the maximal height of an associated prime of I, or
- the maximal analytic spread of I_P, where P runs through the associated primes I.

When I is radical, all these invariants coincide. For more on reductions and a definition of analytic spread, see [SH06, Chapter 8].

Recall that by Exercise 1.30, in order to show that a containment of ideals holds, it is enough to show that the containment holds locally. We will use this idea repeatedly.

With the appropriate tools, the characteristic p statement of Theorem 4.7 for $n = p^e$ turns out to be a beautiful application of the Pigeonhole Principle.

Lemma 4.22 (Hochster-Huneke [HH02])**.** Suppose that I is a radical ideal of big height h in a regular ring R containing a field of prime characteristic p. For all $q = p^e$,

$$I^{(hq)} \subseteq I^{[q]}.$$

Proof. By Exercise 1.30, it is enough to show the containment holds once we localize at the associated primes of $I^{[q]}$. By Lemma 4.20, the associated primes of $I^{[q]}$ coincide with those of I. So let P be an associated prime of I, and note that I_P is generated by at most h elements. Over R_Q, the containment becomes

$$I_Q^{hq} \subseteq I_Q^{[q]}.$$

So consider generators $x_1, \ldots, x_h$ for I_Q. We need to show that

$$(x_1, \ldots, x_h)^{hq} \subseteq (x_1^q, \ldots, x_h^q).$$

Consider $x_1^{a_1} \cdots x_h^{a_h}$ with $a_1 + \cdots + a_h \geqslant hq$. Since $(x_1, \ldots, x_h)^{hq}$ is generated by all such elements, it is enough to show that $x_1^{a_1} \cdots x_h^{a_h} \in (x_1^q, \ldots, x_h^q)$. Since $a_1 + \cdots + a_h \geqslant hq$, the Pidgenhole Principle guarantees that $a_i \geqslant q$ for some i, and thus $x_i^{a_i} \in (x_1^q, \ldots, x_h^q)$. ☐

In fact, the same proof using the full power of the Pidgenhole Principle gives Harbourne's Conjecture 4.33, which we will talk about later, for powers of p, a fact first noted by Craig Huneke:

Exercise 4.23. Suppose that I is a radical ideal of big height h in a regular ring R containing a field of prime characteristic p. Show that for all $q = p^e$,

$$I^{(hq-h+1)} \subseteq I^{[q]} \subseteq I^q.$$

As an easy corollary, we obtain an affirmative answer to Huneke's Question 4.32 in characteristic 2, that is, $I^{(3)} \subseteq I^2$ always holds in characteristic 2.

To prove $I^{(hn)} \subseteq I^n$ holds for all $n \geqslant 1$, we need to use tight closure techniques. The theory of tight closure, developed by Hochster and Huneke, has many important applications across commutative algebra.

Definition 4.24 (Tight Closure)**.** Let R be a domain of prime characteristic p. Given an ideal I in R, the *tight closure* of I is the ideal

$$I^* = \left(z \in A \,|\, \text{ there exists a nonzero } c \in R \text{ such that } cz^q \in I^{[q]} \text{ for all } q = p^e \gg 0\right).$$

Remark 4.25. Notice that $I \subseteq I^*$.

It is sometimes easier to prove something is contained in the tight closure of an ideal than in the ideal itself. This idea is especially useful if we are working over a regular ring, since all ideals are tightly closed.

Theorem 4.26 (Theorem (4.4) in [HH90])**.** Let R be a regular ring containing a field of prime characteristic. Then $I = I^*$ for every ideal I in R.

Theorem 4.27 (Hochster–Huneke, [HH02])**.** Let R be a regular ring of characteristic p, and I be a radical ideal of big height h. Then for all $n \geqslant 1$, $I^{(hn)} \subseteq I^n$.

Proof. Fix n. We will show that if $u \in I^{(hn)}$, then $u \in (I^n)^*$, and since R is regular, that implies that $u \in I^n$. We need to find a nonzero element c I such that $cr^q \in (I^n)^q$ for all $q = p^e$.

Given $q = p^e$, we can write $q = an + r$ for some integers $a, r \geqslant 0$ with $r < n$. Then $u^a \in \left(I^{(hn)}\right)^a \subseteq I^{(han)}$, and

$$I^{hn} u^a \subseteq I^{hr} u^a \subseteq I^{hr} I^{(han)} \subseteq I^{(han+hr)} = I^{(hq)}.$$

Since $I^{(hq)} \subseteq I^{[q]}$, we have $I^{hn} u^a \subseteq I^{[q]}$. Now take powers of n on both sides:

$$I^{hn^2} u^{an} \subseteq \left(I^{[q]}\right)^n = (I^n)^{[q]}.$$

By choice of a, we know $q \geqslant an$, so that

$$I^{hn^2} u^q \subseteq I^{hn^2} u^{an} \subseteq (I^n)^{[q]}.$$

Since R is a domain, there exists a nonzero element $c \in I^{hn^2}$, which does not depend on the choice of q. Such c satisfies $cu^q \in (I^n)^{[q]}$, and thus $u \in I^n$. $\square$

This can be generalized. The following is [ELS01, Theorem 2.2] in the case of smooth complex varieties, and more generally [HH02, Theorem 2.6]:

Theorem 4.28 (Ein–Lazersfeld–Smith, Hochster–Huneke)**.** Let I be a radical ideal of a regular ring containing a field, and let h be the big height of I. Then for all $n \geqslant 1$ and all $k \geqslant 0$, $I^{(hn+kn)} \subseteq \left(I^{(k+1)}\right)^n$.

Exercise 4.29. Show Theorem 4.28 in prime characteristic, essentially by repeating the argument we just gave.

When $k = 0$, this gives $I^{(hn)} \subseteq I^n$. Moreover, in characteristic p, we can obtain a generalized version of the upcoming Harbourne's Conjecture for powers of p:

Lemma 4.30. Let I be a radical ideal in a regular ring R of prime characteristic p and h the big height of I. For all $q = p^e$,

$$I^{(hq+kq-h+1)} \subseteq \left(I^{(k+1)}\right)^{[q]}.$$

Proof. Since containments are local, it is enough the show that the statement holds once we localize at each associated prime of $\left(I^{(k+1)}\right)^{[q]}$. Since

$$\operatorname{Ass}\left(\left(I^{(k+1)}\right)^{[q]}\right) = \operatorname{Ass}\left(I^{(k+1)}\right) = \operatorname{Ass}(I),$$

it is enough to show that the statement holds at the associated primes of I. If P is an associated prime of I, the statement we need to show over R_P is the following:

$$P_P^{hq+kq-h+1} \subseteq \left(P_P^{[q]}\right)^{k+1}.$$

The claim now follows by a similar argument as Lemma 4.22. Full details in [HH02, Lemma 2.4 (a)]. $\square$

4.3 Harbourne's Conjecture

The containments provided by Theorem 4.7 are not necessarily best possible.

Example 4.31. The ideal $I = (x, y) \cap (x, z) \cap (y, z)$ from Example 2.2 has big height 2, so that Theorem 4.7 implies that $I^{(2n)} \subseteq I^n$ for all $n \geqslant 1$. However, $I^{(3)} \subseteq I^2$, even though the theorem only guarantees $I^{(4)} \subseteq I^2$.

Question 4.32 (Huneke, 2000)**.** Let P be a prime ideal of height 2 in a regular local ring containing a field. Does the containment $P^{(3)} \subseteq P^2$ always hold?

This question remains open even in dimension 3. Harbourne proposed the following generalization of Question 4.32, which can be found in [HH13, BRH+09]:

Conjecture 4.33 (Harbourne, 2006)**.** Let I be a radical homogeneous ideal in $k[\mathbb{P}^N]$, and let h be the big height of I. Then for all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Remark 4.34. Equivalently, Harbourne's Conjecture asks if $I^{(n)} \subseteq I^{\lceil \frac{n}{h} \rceil}$ for all $n \geqslant 1$.

Remark 4.35. When $h = 2$, the conjecture asks that $I^{(2n-1)} \subseteq I^n$, and in particular that $I^{(3)} \subseteq I^2$.

As we saw before, the containment $I^{(hn-h+1)} \subseteq I^n$ does hold if we take $n = p^e$ in characteristic p. For any n, there are various cases where this conjecture is known to hold:

- If I is a monomial ideal (which first appeared in [BRH+09, Example 8.4.5]);
- if I corresponds to a generic set of points in $\mathbb{P}^2$ ([BH10]) or $\mathbb{P}^3$ ([Dum15]);
- if I corresponds to a star configuration of points ([HH13]),

among others. We will see that the conjecture also holds if I defines an F-pure ring, which in particular recovers the result for monomial ideals.

Exercise 4.36. Let I be a squarefree monomial ideal. Show that I satisfies Harbourne's Conjecture.

Hint: given a monomial ideal, we can take its bracket power, which is a sort of fake Frobenius power. The n-th bracket power of the monomial ideal I is the ideal

$$I^{[n]} = \left(f^n \,\middle|\, f \in I \text{ is a monomial}\right).$$

As the notation suggests, these behave a lot like the Frobenius powers.

Unfortunately, Conjecture 4.33 turns out to be too general; it does not hold for all homogeneous radical ideals.

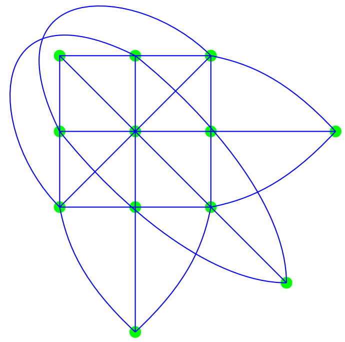

Figure 1: Fermat configuration of points when $n = 3$.

Example 4.37 (Fermat configurations of points)**.** Let $n \geqslant 3$ be an integer and consider a field k of characteristic not 2 such that k contains n distinct roots of unity. Let $R = k[x, y, z]$, and consider the ideal

$$I = (x(y^n - z^n), y(z^n - x^n), z(x^n - y^n)) .$$

When $n = 3$, this corresponds to a configuration of 12 points in $\mathbb{P}^2$, as described in Figure 1. Over $\mathbb{P}^2(\mathbb{C})$, these 12 points are given by the 3 coordinate points plus the 9 points defined by the intersections of $y^3 - z^3$, $z^3 - x^3$ and $x^3 - y^3$.

The ideal I is radical and has pure height 2. However, $I^{(3)} \nsubseteq I^2$, since the element $f = (y^n - z^n)(z^n - x^n)(x^n - y^n) \in I^{(3)}$ but not in I^2. This can be shown via geometric arguments, noting that f defines 9 lines, some three of which go through each of the 12 points.

This was first proved by Dumnicki, Szemberg e Tutaj-Gasińska [DSTG13] over $k = \mathbb{C}$, and then generalized in [HS15, Proposition 3.1] to any k and any n. Other extensions of this example can be found in [Dra17, MS18b].

Other configurations of points in $\mathbb{P}^2$ have been shown to produce ideals that fail the containment $I^{(3)} \subseteq I^2$, such as the Klein and Wiman configurations of points [Sec15]. Given a configuration of points in $\mathbb{P}^k$ that produces an ideal I with $I^{(hn-h+1)} \nsubseteq I^n$, one can produce other counterexamples to the same type of containment by applying flat morphisms $\mathbb{P}^k \to \mathbb{P}^k$; see the work of Solomon Akesseh [Ake17].

Example 4.38. Harbourne and Seceleanu [HS15] showed that $I^{(hn-h+1)} \subseteq I^n$ can fail for arbitrarily high values of n in characteristic $p > 0$. However, their counterexamples are constructed depending on n, meaning that given n, there exists an ideal I_n of pure height 2 (corresponding, once more, to a configuration of points in $\mathbb{P}^2$) which fails $I_n^{(hn-h+1)} \subseteq I_n^n$.

However, even these examples satisfy the following open conjecture:

Conjecture 4.39 (Harbourne stable [Gri])**.** If I is a radical ideal of big height h in a regular ring, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \gg 0$.

We are asking if Harbourne's Conjecure holds for n large — where large enough should depend on I, as Harbourne and Seceleanu's examples suggest [HS15]. There are no counterexamples known to this conjecture.

Também não conhecemos nenhum contra-exemplo *primo* para a Conjectura de Harbourne. Em particular, é possível que a resposta à pergunta de Huneke seja afirmativa, e que $P^{(3)} \subseteq P^2$ para ideais primos P num anel de séries de potências.

4.4 Harbourne's Conjecture in characteristic p

We will show that Harbourne's Conjecture always holds for I when R/I is a nice enough ring: we will ask that R/I be F-pure.

Definition 4.40 (F-finite ring)**.** Let A be a noetherian ring of prime characteristic p. We say that A is *F-finite* if A is a finitely generated module over itself via the action of the Frobenius map.

Definition 4.41. If A is F-finite and reduced, the ring of p^e-roots of A is denoted by $F^e_* A$, and the inclusion $A \hookrightarrow F^e_* A$ can be identified with F^e. The fact that A is F-finite implies that $F^e_* A$ is a finitely generated module over A for all $q = p^e$.

Example 4.42. If k is a perfect field, then $k[x_1, \ldots, x_n]$ is F-finite. In fact, every k algebra essentially of finite type over k is F-finite.

We will study F-pure rings, which were introduced by Hochster and Roberts in [HR76].

Definition 4.43 (F-pure ring)**.** Let A be a noetherian ring of prime characteristic p. We say that A is *F-pure* if for any A-module M, $F \otimes 1 \colon A \otimes M \longrightarrow A \otimes M$ is injective.

Definition 4.44 (F-split ring)**.** Let A be a noetherian ring of prime characteristic p. We say that A is *F-split* if the inclusion $R \hookrightarrow F^e_* R$ splits for every (equivalently, some) $q = p^e$, that is, if there exists a homomorphismo of R-módulos $F^e_* R \hookrightarrow R$ such that the composition

$$\begin{tikzcd} R \arrow[r] & F^e_* R \arrow[l, bend left, dashed] \end{tikzcd}$$

is the identity on R.

Lemma 4.45. If A is F-finite, then A is F-pure if and only A is F-split.

Proof. See [HR76, Corollary 5.3]. □

The following theorem characterizes ideals that define F-pure rings over a regular ring:

Theorem 4.46 (Fedder's Criterion for F-purity, Theorem 1.12 in [Fed83])**.** Let $(R, \mathfrak{m})$ be a regular local ring of prime characteristic p. Given an ideal I in R, R/I is F-pure if and only if for all $q = p^e \gg 0$,

$$\left(I^{[q]} : I\right) \nsubseteq \mathfrak{m}^{[q]}.$$

One particularly nice thing about this criterion is that it is also sufficient to test $\left(I^{[p]} : I\right) \nsubseteq \mathfrak{m}^{[p]}$. This is something Macaulay2 [GS] can test when R is a polynomial ring over a field of characteristic p, where we take for $\mathfrak{m}$ the ideal generated by all the variables.

Here are some examples of F-pure rings:

- Regular rings are always F-pure.
- If I is a squarefree monomial ideal in a polynomial ring over a field, then R/I is an F-pure ring. (Exercise!)
- Veronese rings of polynomial rings are F-pure: the k-algebra generated by all the monomials in v variables and a fixed degree d.
- Generic determinantal rings are F-pure.

We are now ready to show that if R is a regular ring and R/I is F-pure, then I satisfies Harbourne's Conjecture. First, we record the result we are trying to prove.

Theorem 4.47 (Theorem 3.3 in [GH19])**.** Let R be a regular ring of prime characteristic p. Let I be an ideal in R such that R/I is F-pure, and let h be the big height of I. Then for all $n \geqslant 1$, $I^{(hn-h+1)} \subseteq I^n$.

This can be extended to some singular settings: if R is an F-finite Gorenstein, this still holds for ideals of finite projective dimension [GMS19].

Naively, the idea of the proof is to study the colon ideal $\left(I^n : I^{(hn-h+1)}\right)$. The colon ideal $(J : I)$ measures the failure of $I \subseteq J$, and $(J : I) = R$ precisely when $I \subseteq J$. In order to show that $\left(I^n : I^{(hn-h+1)}\right) = R$, we need to show that this ideal contains some *large* ideal; Fedder's Criterion 4.46 provides the perfect candidate. The proof in [GH19] does just that — we show that

$$\left(I^{[q]} : I\right) \subseteq \left(II^{(n)} : I^{(n+h)}\right)^{[q]},$$

for all ideals I and all $q = p^e \gg 0$, and when R/I is F-pure that implies Harbourne's Conjecture. The proof we will follow here uses the same techniques, but instead we will show a slightly more powerful lemma.

Lemma 4.48. Let R be a regular ring of prime characteristic p. Let I be a radical ideal in R and h the big height of I. For all $n \geqslant 1$,

$$\left(I^{[q]} : I\right) \subseteq \left(II^{(n)} : I^{(n+h)}\right)^{[q]}$$

for all $q = p^e \gg 0$.

Proof. Recall that

$$\left(II^{(n)} : I^{(n+h)}\right)^{[q]} = \left(\left(II^{(n)}\right)^{[q]} : \left(I^{(n+h)}\right)^{[q]}\right),$$

by Lemma 4.19. Take $s \in \left(I^{[q]} : I\right)$. Then $sI^{(n+h)} \subseteq sI \subseteq I^{[q]}$, so

$$s\left(I^{(n+h)}\right)^{[q]} \subseteq \left(sI^{(n+h)}\right)\left(I^{(n+h)}\right)^{q-1} \subseteq I^{[q]}\left(I^{(n+h)}\right)^{q-1}.$$

We will show that

$$\left(I^{(n+h)}\right)^{q-1} \subseteq \left(I^{(n)}\right)^{[q]},$$

which implies that

$$s\left(I^{(n+h)}\right)^{[q]} \subseteq \left(II^{(n)}\right)^{[q]},$$

completing the proof.

Notice that, by Lemma 1.21,

$$\left(I^{(n+h)}\right)^{q-1} \subseteq I^{((n+h)(q-1))}.$$

By Lemma 4.30 with $k = n-1$, we obtain the following containment:

$$I^{(hq+(n-1)q-h+1)} \subseteq \left(I^{(n)}\right)^{[q]}.$$

We claim that for all $q \gg 0$, $\left(I^{(n+h)}\right)^{q-1} \subseteq I^{(hq+(n-1)q-h+1)}$, which would conclude the proof that $\left(I^{(n+h)}\right)^{q-1} \subseteq \left(I^{(n)}\right)^{[q]}$. To show that the claim, it is enough to prove that

$$(n+h)(q-1) \geqslant hq + (n-1)q - h + 1$$

for large values of q. This can be seen by comparing the coefficients in q, and noticing that $n+h \geqslant n+h-1$, or by explicitly solving the inequality. In particular, it holds as long as $q \geqslant n+1$. $\square$

Corollary 4.49. Let R be a regular ring of prime characteristic p. Let I be an ideal in R with R/I F-pure, and let h be the big height of I. Then for all $n \geqslant 1$,

$$I^{(n+h)} \subseteq II^{(n)}.$$

Proof. First, note that we can reduce to the local case: containments are local statements, by Exercise 1.30; the big height of an ideal does not increase under localization; and all localizations of an F-pure ring are F-pure [HR74, 6.2]. So suppose that $(R, \mathfrak{m})$ is a regular local ring, and that R/I is F-pure.

Fix $n \geqslant 1$, and consider q as in Lemma 4.48. Then for all $q \gg 0$,

$$\left(I^{[q]} : I\right) \subseteq \left(II^{(n)} : I^{(n+h)}\right)^{[q]}.$$

If $I^{(n+h)} \nsubseteq II^{(n)}$, then $\left(II^{(n)} : I^{(n+h)}\right)^{[q]} \subseteq \mathfrak{m}^{[q]}$, contradicting Fedder's Criterion. $\square$

We can now show that Harbourne's conjecture holds for ideals defining F-pure rings.

Exercise 4.50. Prove Theorem 4.47 using Corollary 4.49. That is, show that if R is a regular ring of characteristic p and R/I is F-pure, then I satisfies Harbourne's Conjecture.

It is natural to ask if we can improve the answer to the containment problem given by Theorem 4.47. One way to do that would be to show that $I^{(hn-h)} \subseteq I^n$ for all $n \geqslant 1$, which unfortunately does not hold for all ideals defining F-pure rings.

Exercise 4.51. Let $R = k[x_1, \ldots, x_d]$ and consider the squarefree monomial ideal

$$I = \bigcap_{i<j} (x_i, x_j).$$

Show that while $I^{(2n-1)} \nsubseteq I^n$ holds for all $n \geqslant 1$, $I^{(2n-2)} \nsubseteq I^n$ for $n < d$. What happens when $n = d$? How does this example generalize to higher height?

But in fact, Corollary 4.49 implies more than just Harbourne's Conjecture.

Exercise 4.52. Let R be a regular ring of prime characteristic p, and consider an ideal I in R such that R/I is F-pure. Show that given any integer $c \geqslant 1$, if $I^{(hk-c)} \subseteq I^k$ for some n, then $I^{(hn-c)} \subseteq I^n$ for all $n \gg 0$.

When R/I is strongly F-regular, we can improve this.

Definition 4.53 (Strongly F-regular ring)**.** A Noetherian reduced F-finite ring R is called *strongly F-regular* if for every $c \in R$ that is not in any minimal prime of R, there exists $e \gg 0$ such that the map $R \to R^{1/p^e}$ sending 1 to c^{1/p^e} splits as a map of R-modules.

Determinantal rings and Veronese rings are examples of strongly F-regular rings.

Theorem 4.54 (Theorem 4.1 in [GH19])**.** Let R be a regular F-finite ring of prime characteristic p, and consider an ideal I in R of big height h such that R/I is strongly F-regular. Then for all $n \geqslant 1$,

$$I^{((h-1)n+1)} \subseteq I^{n+1}.$$

In particular, when $h = 2$ we have $I^{(n)} = I^n$ for all $n \geqslant 1$.

This theorem essentially says that if R/I is strongly F-regular, then I satisfies a version of Harbourne's Conjecture where we can replace the big height h of I by $h - 1$.

The result follows from a Fedder-like criterion for strong F-regularity together with the following lemma [GH19, Lemma 3.2]:

Lemma 4.55. Let R be a regular ring of prime characteristic p, I an ideal in R, and $h \geqslant 2$ the maximal height of a minimal prime of I. Then for all $d \geqslant h - 1$ and for all $q = p^e$,

$$\left(I^d : I^{(d)}\right)\left(I^{[q]} : I\right) \subseteq \left(I I^{(d+1-h)} : I^{(d)}\right)^{[q]}.$$

The Fedder-like criterion we need was first shown by Donna Glassbrenner:

Theorem 4.56 (Glassbrenner's Criterion for strong F-regularity [Gla96])**.** Let $(R, \mathfrak{m})$ be an F-finite regular local ring of prime characteristic p. Given a proper radical ideal I of R, R/I is strongly F-regular if and only if for each element $c \in R$ not in any minimal prime of I, $c\left(I^{[q]} : I\right) \nsubseteq \mathfrak{m}^{[q]}$ for all $q = p^e \gg 0$.

Exercise 4.57. Let I be an ideal in a noetherian ring. Show that $\left(I^d : I^{(d)}\right)$ always contains an element that is not in any minimal prime of I.

Exercise 4.58. Prove Theorem 4.54 using Lemma 4.55.

Exercise 4.59. What does Theorem 4.54 say for primes of height 2? Find examples of such primes that are not complete intersections.

5 Historical remarks

Symbolic powers first arose from the theory of primary decomposition. In 1905, world chess champion Emanuel Lasker [Las05] showed that every ideal in a polynomial or power series ring over a field has a primary decomposition, and in 1921, Emmy Noether [Noe21] extended Lasker's result to any noetherian ring.

In 1949, Zariski gave the first proof of what is now known as the Zariski–Nagata theorem [Zar49, Nag62]. This result has been extended by Eisenbud and Hochster [EH79], and it can be phrased in terms of differential operators [DDSG$^+$18, **?**, **?**]. Morally, Zariski–Nagata says that the symbolic powers have nice geometric properties, while the ordinary algebraic powers have no precise geometric meaning. On the other hand, $P^{(n)}$ can be extremely difficult to compute algebraically, while determining P^n from P is fairly simple.

In the 1970s and 1980s, there was a lot of interest in comparing the topologies determined by the ordinary and symbolic powers. The fact that two topologies are equivalent for a given ideal I amounts to saying that for all positive integers b, there exists a value a such that $I^{(a)} \subseteq I^b$. Schenzel asked if this should imply that there exists a constant k such that $I^{(kn)} \subseteq I^n$; in 2000, Irena Swanson [Swa00] answered this question positively.

Soon after Swanson's theorem, Ein, Lazarsfeld, and Smith [ELS01] determined what this constant k is over an affine variety of characteristic 0, and Hochster and Huneke [HH02] generalized the result for the case of a regular ring containing a field and any ideal I. Very recently, Ma and Schwede [MS18a] have settled the mixed characteristic case. Given a radical ideal I, the constant k given by Swanson's thereom can be taken to be the big height of I, an invariant depending only on the associated primes of I. In particular, these results imply that $I^{(dn)} \subseteq I^n$ for all $n \geqslant 1$, where the constant d can be taken to be independent of the choice of ideal. Whether such a uniform result holds for prime ideals in a more general setting is still an open question. However, this has been settled in some cases [HKV09, HKV15, Wal18b, Wal16, Wal18a].

In the 1980s and 1990s, much effort was devoted to studying symbolic Rees algebras, $\oplus I^{(n)}t^n$, especially for prime ideals. The main motivation was a question raised by Cowsik in the 1980s: should the symbolic Rees algebra always be noetherian, in particular for prime ideals P in a regular local ring R such that $\dim(R/P) = 1$? Cowsik's motivation was a result of his [Cow84] showing that a positive answer would imply that all such primes are set-theoretic complete intersections, that is, complete intersections up to radical. Eliahou, Huckaba, Huneke, Vasconcelos and others proved various criteria that imply noetherianity. However, in 1985, Paul Roberts [Rob85] answered Cowsik's question negatively. On the other hand, space monomial curves were known to be set-theoretic complete intersections [Bre79, Her80, Val81], and much work was devoted to studying their symbolic Rees algebras. Surprisingly, the answer to Cowsik's question is negative even for this class of primes, with the first non-noetherian example found by Morimoto and Goto [GM92]. Cutkostky [Cut91] also gave criteria to determine whether the symbolic Rees algebra of a given space monomial curve is noetherian.

Besides being an interesting subject in its own right, symbolic powers appear as auxiliary tools in several important results in commutative algebra, such as Krull's Principal Ideal Theorem, Chevalley's Lemma, or in giving a proof in prime characteristic for the fact that regular local rings are UFDs. Hartshorne's proof of the Hartshorne—Lichtenbaum Vanishing

Theorem also makes use of symbolic powers. Explicitly, Hartshorne's proof of this local cohomology result uses the fact that certain symbolic and -adic topologies are equivalent, and thus local cohomology can be computed using symbolic powers.

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