Problem Set 5
solutions

Problem 1. Let $f\colon G \to H$ be a group homomorphism. Show that $\ker f$ is a normal subgroup of G.

Proof. We already know that $\ker f$ is a subgroup of G, so we only need to prove normality. Consider $g \in \ker f$, and any $h \in G$. Then

$$\begin{aligned} f(hgh^{-1}) &= f(h)f(g)f(h)^{-1} && \text{since } f \text{ is a homomorphism} \\ &= f(h)f(h)^{-1} && \text{since } f(g) = e_H \\ &= e_H, \end{aligned}$$

so $hgh^{-1} \in H$. We conclude that H is normal. □

Problem 2. Let H and K be normal subgroups of a group G such that $H \cap K = \{e\}$. Prove that $xy = yx$ for all $x \in H, y \in K$.

Proof. Let $h \in H$ and $k \in K$. As H is normal, $k^{-1}hk \in H$. Hence, $h^{-1}k^{-1}hk \in H$, since h is also an element of H. Similarly, as K is normal, $h^{-1}k^{-1}h \in K$, so $h^{-1}k^{-1}hk \in K$. Thus

$$h^{-1}k^{-1}hk \in H \cap K = \{e\},$$

so $hk = kh$. □

Problem 3. Let $f\colon G \to H$ be a group homomorphism.

(3.1) Prove that if $K \trianglelefteq H$ then the preimage $f^{-1}(K)$ of K is a normal subgroup of G.

Proof. The fact that the preimage of a subgroup is a subgroup was proven on the previous homework (and hence can be used without proof here). We justify the normality of the preimage. Let $g \in G$ and $\ell \in f^{-1}(K)$. Then $f(\ell) \in K$ and $f(g\ell g^{-1}) = f(g)f(\ell)f(g)^{-1} \in K$ by the normality of K. Therefore $g\ell g^{-1} \in f^{-1}(K)$ for all $g \in G$ and so $gf^{-1}(K)g^{-1} \subseteq f^{-1}(K)$ for all $g \in G$. By a proposition from class this suffices to prove that $f^{-1}(K) \trianglelefteq G$. □

(3.2) Give an example showing that if $L \trianglelefteq G$ then $f(L)$ need not be a normal subgroup of H.

Proof. There are many different correct answers. Here is one.

Consider $G = \{\mathrm{id}_{[3]}, (12)\}$ and $H = S_3$ and let f be the inclusion map of G in H. Then f is a group homomorphism since it is a restriction of the identity map, which is a group homomorphism. Moreover, $\mathrm{im}(f) = \{\mathrm{id}_{[3]}, (12)\}$. The mage of f is not a normal subgroup of S_3 since $(13)(12)(13)^{-1} = (23) \notin \mathrm{im}(f)$. □

Problem 4. Let G be a group, S a subset of G, and $H = \langle S \rangle$.

(4.1) Prove that $H \trianglelefteq G$ if and only if $gsg^{-1} \in H$ for every $s \in S$ and $g \in G$.

Proof. Suppose $gsg^{-1} \in H$ for all $s \in S$. Let $h \in H$ and $g \in G$. Then $h = s_1^{e_1} s_2^{e_2} \cdots s_n^{e_n}$ for some $s_1, \dots, s_n \in S$ and $e_1, \dots, e_n \in \{\pm 1\}$. Let $g \in G$. Then

$$ghg^{-1} = g(s_1^{e_1} s_2^{e_2} \cdots s_n^{e_n})g^{-1} = (gs_1^{e_1}g^{-1})(gs_2^{e_2}g^{-1}) \cdots (gs_n^{e_n}g^{-1}).$$

Note that if $e_i = -1$ then $gs_i^{-1}g^{-1} = (gs_ig^{-1})^{-1} \in H$. Thus, $gs_i^{e_i}g^{-1} \in H$ for all i, and hence $ghg^{-1} \in H$. Therefore, $H \triangleleft G$. The reverse implication is true by definition of normal subgroup. □

(4.2) Consider the commutator subgroup of G

$$[G, G] := \langle aba^{-1}b^{-1} \mid a, b \in G \rangle$$

generated by all the commutators of elements in G. Prove that $[G, G] \trianglelefteq G$.

Proof. Let $g \in G$ and $s = aba^{-1}b^{-1}$. Set $x = gag^{-1}$ and $y = gbg^{-1}$, and note that

$$gsg^{-1} = xyx^{-1}y^{-1} \in S \subseteq H.$$

Hence, $H \trianglelefteq G$ by (4.1). □

Problem 5. Show that any subgroup of index two is normal. This means: show that if G is a group, H is a subgroup and $[G : H] = 2$, i.e the number of left (or right) cosets of H in G is two, then H is normal.

Proof. Since $[G : H] = 2$, there are exactly two left cosets and two right cosets of H in G. The set $H = e_G H = H e_G$ is both a left coset and a right coset. We will show that for any $g \in G$, either

$$\begin{cases} gH = H = Hg & \text{if } g \in H \\ gH = G \setminus H = Hg & \text{if } g \notin H. \end{cases} \tag{1}$$

Indeed, if $g \in H$ then

$$e_G^{-1} g \in H \quad \text{so} \quad gH = e_G H = H$$

and

$$g e_G^{-1} \in H \quad \text{so} \quad Hg = He_G = H.$$

Thus

$$gH = Hg$$

when $g \in H$.

If $g \notin H$ then $e_G^{-1} g \notin H$ so $gH \neq H$. Since there are only two left cosets for H in G, those two left cosets must be H and gH. Since these cosets partition G, we must have $gH = G \setminus H$.

Similarly for right cosets: if $g \notin H$ then $g e_G^{-1} \notin H$ so $Hg \neq H$. Since there are only two right cosets for H in G, they must be H and Hg. Since these cosets partition G, we have $Hg = G \setminus H$. We conclude that $gH = Hg$ for all $g \notin H$.

Since $gH = Hg$ for all $g \in G$, it follows by a result proven in class that $H \trianglelefteq G$. □

Problem 6. Let G be any group. Show that if $G/\,\mathrm{Z}(G)$ is cyclic, then G is abelian.

Proof. Let $Z := \mathrm{Z}(G)$ and suppose $G/Z = \langle xZ \rangle$ for some $x \in G$. Let $a, b \in G$. Then $aZ = x^i Z$ and $bZ = x^j Z$ for some i, j. Hence, $a = x^i z_1$ and $b = x^j z_2$ for some $z_1, z_2 \in G$. Then

$$ba = (x^j z_2)(x^i z_1) = x^{j+i} z_1 z_2 = (x^i z_1)(x^j z_2) = ab.$$

Therefore, G is abelian. □