Midterm

Instructions: You may use any results proved in class or in the problem sets, except for the specific question being asked. This includes any formulas about elements of S_n or D_n we proved before. You should clearly state any facts you are using. You are also allowed to use anything stated in one problem to solve a different problem, even if you have not yet proved it. Remember to show all your work, and to write clearly and using complete sentences. No calculators, notes, cellphones, smartwatches, or other outside assistance allowed.

I. Short questions

Problem 1. State the First Isomorphism Theorem for groups.

Problem 2. For each of the questions below, give an example with the required properties. No explanations required.

(a) A group that is not cyclic.
(b) A group G that is not abelian and a subgroup H of G that is abelian.

Problem 3. For each of the questions below, give an example with the required properties, and briefly explain why the required properties are satisfied.

(a) A group G and a subgroup H that is *not* normal in G.
(b) Two groups G and H of the same finite order that are *not* isomorphic.

Problem 4. Find (with proof!) the order of $(23)(5689)$ in S_{10}.
You can freely use any results we proved about orders of elements in S_n; avoid difficult calculations!

II. Old problems

Choose **2** of the following problems.

Problem 5. Let $f\colon G \to H$ be a group homomorphism. Show that $\ker f$ is a normal subgroup of G. Note: You must show *both* that $\ker f$ is a subgroup of G and that it is normal.

Problem 6. Let G be a group and H and H' subgroups of G. Prove that $H \cup H'$ is a subgroup of G if and only if $H \subseteq H'$ or $H' \subseteq H$.

Problem 7. Let G be any group. Show that if $G/\operatorname{Z}(G)$ is cyclic, then G is abelian.

III. New problems

Choose any **2** of the following problems.

Problem 8. Let G be a group (not necessarily finite) such that every element $g \in G$ satisfies $g^2 = e$. Show that G is abelian.

Problem 9. In this problem, you can use without proof that every finite cyclic group of order n is isomorphic to $\mathbb{Z}/n$.

(a) Show that any group of prime order p is cyclic.
(b) Now suppose that G is any nontrivial group, not necessarily finite. Show that G has no nontrivial proper subgroups if and only if G is finite of order p, where p is prime.

Problem 10. Show that there is no surjective group homomorphism $f\colon S_5 \longrightarrow S_4$.