# Some old qualifying exam questions

Here are some old qualifying exam problems you are already ready to solve.

**Problem 1** (January 2024)**.** Let $H$ be a subgroup of a group $G$. Show that $[G, G] \leq H$ if and only if $H$ is a normal subgroup of $G$ and $G/H$ is abelian.

**Problem 2** (June 2023)**.** Let $G$ be a group and let $H$ be a subgroup of $G$. The following sets are subgroups of $G$

$$N_G(H) = \{g \in G : gHg^{-1} \subseteq H\} \quad \text{and} \quad C_G(H) = \{g \in G : gh = hg, \forall h \in H\},$$

a fact which you may use without proof. Prove that $C_G(H)$ is a normal subgroup of $N_G(H)$.

Note: Above I wrote only part (a); while you know what you need for part (b), that is a problem best left for later in the semester.

**Problem 3** (June 2023)**.** Let $G$ be a group with center $\mathrm{Z}(G)$. Prove that if the quotient group $G/\mathrm{Z}(G)$ is cyclic, then $G$ is abelian.

**Problem 4** (May 2021)**.** Let $G$ be a group (not necessarily finite) and $H$ a nonempty subset of $G$ that is closed under multiplication. Suppose that for all $g \in G$ we have $g^2 \in H$.

(a) Show that $H$ must be a subgroup of $G$.

(b) Show that $H$ must be a normal subgroup of $G$.

(c) Show that $G/H$ must be abelian.