

Midterm solutions

I. Short questions

Problem 1. State the First Isomorphism Theorem for groups.

Theorem 1. Let $f: G \rightarrow H$ be a group homomorphism. Then $\ker f \trianglelefteq G$ and $G/\ker f \cong \text{im } f$.

Problem 2. For each of the questions below, give an example with the required properties. No explanations required.

- (a) A group that is not cyclic.

Solution 1. There are many correct answers.

Here are some examples: D_4 (or D_n for any $n \geq 3$), $\mathbb{Z}_2 \times \mathbb{Z}_2$.

- (b) A group G that is not abelian and a subgroup H of G that is abelian.

Solution 2. D_4 and the subgroup $\{e, s\}$.

Problem 3. For each of the questions below, give an example with the required properties, and briefly explain why the required properties are satisfied.

- (a) A group G and a subgroup H that is *not* normal in G .

Solution 3. Let $G = D_4$ and consider the subgroup $H = \{e, s\}$. Note that H is not a normal subgroup, since we showed in class that $srs = r^{-1} = r^3$, and since $s = s^{-1}$ we have

$$rs = sr^{-1} \implies rsr^{-1} = sr^{-1}r^{-1} = sr^{-2}.$$

- (b) Two groups G and H of the same finite order that are *not* isomorphic.

Solution 4. The groups \mathbb{Z}_4 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ both have order 4 but are not isomorphic, since \mathbb{Z}_4 is cyclic and thus has an element of order 4, while every nontrivial element of $\mathbb{Z}_2 \times \mathbb{Z}_2$ has order 2.

Problem 4. Find (with proof!) the order of $(23)(5689)$ in S_{10} .

You can freely use any results we might have proved about this.

Solution 5. We proved in class that k -cycles have order k , so $|(23)| = 2$ and $|(5689)| = 4$. We also showed in a problem set that if σ and τ are disjoint cycles, then $|\sigma\tau| = \text{lcm}(|\sigma|, |\tau|)$, and since (23) and (5689) are disjoint cycles, we conclude that

$$|(23)(5689)| = \text{lcm}(|(23)|, |(5689)|) = \text{lcm}(2, 4) = 4.$$

II. Old problems

Choose **2** of the following problems.

Problem 5. Let $f: G \rightarrow H$ be a group homomorphism. Show that $\ker f$ is a normal subgroup of G . Note: You must show *both* that $\ker f$ is a subgroup of G and that it is normal.

Proof. First, we need to show that $\ker f$ is a subgroup of G . Since $f(e) = e$, we have $e \in \ker f$, and in particular $\ker f$ is nonempty. Moreover, given $g, h \in \ker(f)$,

$$f(gh^{-1}) = f(g)f(h)^{-1} = ee^{-1} = e,$$

so $gh^{-1} \in \ker f$. By the One-Step Test, $\ker f$ is a subgroup of G .

Now we need to show this is a normal subgroup of G . Given any $g \in G$ and any $h \in \ker f$,

$$f(ghg^{-1}) = f(g)f(h)f(g)^{-1} = f(g) \cdot e \cdot f(g)^{-1} = f(g)f(g)^{-1} = e.$$

Therefore, $ghg^{-1} \in \ker f$, and we conclude that $\ker f$ is a normal subgroup of G . \square

Problem 6. Let G be a group and H and H' subgroups of G . Prove that $H \cup H'$ is a subgroup of G if and only if $H \subseteq H'$ or $H' \subseteq H$.

Proof. (\Leftarrow) If $H \subseteq H'$ or $H' \subseteq H$, then we either have $H \cup H' = H$ or $H \cup H' = H'$, which are both subgroups.

(\Rightarrow) Suppose by way of contradiction that $H \cup H'$ is a subgroup but $H \not\subseteq H'$ and $H' \not\subseteq H$. Choose $a \in H \setminus H'$ and $b \in H' \setminus H$. Then $a, b \in H \cup H'$, and since $H \cup H'$ must be closed for the multiplication, we conclude that $ab \in H \cup H'$. But if $ab \in H$, then multiplying on the left by a^{-1} gives $b = a^{-1}(ab) \in H$, a contradiction. A similar contradiction holds if $ab \in H'$, giving us $a = (ab)b^{-1}$. Thus $ab \notin H \cup H'$. \square

Problem 7. Let G be any group. Show that if $G/Z(G)$ is cyclic, then G is abelian.

Proof. Let $Z := Z(G)$ and suppose $G/Z = \langle xZ \rangle$ for some $x \in G$. Let $a, b \in G$. Then $aZ = x^i Z$ and $bZ = x^j Z$ for some i, j . Hence, $a = x^i z_1$ and $b = x^j z_2$ for some $z_1, z_2 \in G$. Then

$$ba = (x^j z_2)(x^i z_1) = x^{j+i} z_1 z_2 = (x^i z_1)(x^j z_2) = ab.$$

Therefore, G is abelian. \square

III. New problems

Choose any **2** of the following problems.

Problem 8. Let G be a group (not necessarily finite) such that every element $g \in G$ satisfies $g^2 = e$. Show that G is abelian.

Proof. Given any $g \in G$, since $g^2 = e$ then $g = g^{-1}$. Let $x, y \in G$. We have $x^{-1} = x$, $y^{-1} = y$, and $(yx)^{-1} = yx$. Moreover,

$$\begin{aligned} xy &= x^{-1}y^{-1} \quad \text{since } x = x^{-1} \text{ and } y = y^{-1} \\ &= (yx)^{-1} \\ &= yx \quad \text{since } (yx)^{-1} = yx. \end{aligned}$$

Thus $xy = yx$, and G is abelian. \square

Problem 9. In this problem, you can use without proof that every finite cyclic group of order n is isomorphic to \mathbb{Z}/n .

- (a) Show that any group of prime order p is cyclic, and thus it must be isomorphic to \mathbb{Z}/p .

Proof. Suppose that G is a group of order p , where p is prime. Since $p \geq 2$, then G has at least one nontrivial element $g \in G$. By Lagrange's Theorem, the order of g must divide the order of G , but $|G|$ is prime and $|g| \neq 1$, so $|g| = |G| = p$. But $|\langle g \rangle| = |g| = p$, so $\langle g \rangle = G$. We conclude that G is cyclic and generated by g . Since every cyclic group of order p is isomorphic to \mathbb{Z}/p , we conclude that $G \cong \mathbb{Z}/p$. \square

- (b) Now suppose that G is any nontrivial group, not necessarily finite. Show that G has no nontrivial proper subgroups if and only if G is finite of order p , where p is prime.

Proof. (\Leftarrow) Suppose that G is a finite group of order p . We showed in part (a) that $G \cong \mathbb{Z}/p$. Let us write $[i]$ for the class of i in \mathbb{Z}/p .

Suppose that H is any nontrivial subgroup of G . Then H contains some nontrivial element of G , say $[i]$, where we can take $1 \leq i < p$. In particular, $\gcd(i, p) = 1$, so there exist integers a, b such that $ai + bp = 1$, and thus

$$[1] = a[i] \in \langle [a] \rangle \subseteq H.$$

We conclude that $H = G$.

(\Rightarrow) Suppose that G is nontrivial but has no nontrivial subgroups. Let $g \in G$ be any nontrivial element in G . Then $\langle g \rangle$ is a nontrivial subgroup of G , and so we must have $\langle g \rangle = G$. In particular, G is cyclic. Now if G is infinite, then g must have infinite order. We showed in a problem set that all the powers g^n with $n \geq 0$ are distinct; in particular, $g \notin \langle g^2 \rangle$, so $\langle g^2 \rangle$ is a proper nontrivial subgroup of G . Thus G must be finite.

So we have shown that G must be a finite cyclic group, and thus $G \cong \mathbb{Z}/n$ for some n . If n is not prime, then we can write $n = ab$ for some integers $1 < a, b < n$. In particular, $b[a] = [0]$, and thus $[a]$ has order at most b , and $|\langle [a] \rangle|$ has at most b elements, so $\langle [a] \rangle \neq G$. On the other hand, $\langle [a] \rangle$ is nontrivial since it contains $[a] \neq [0]$. Thus $\langle [a] \rangle$ is a proper nontrivial subgroup. \square

Problem 10. Show that there is no surjective group homomorphism $f: S_5 \rightarrow S_4$.

Proof 1. Suppose that $f: S_5 \rightarrow S_4$ is any group homomorphism, and let σ be a 5-cycle. The σ has order 5, and thus

$$f(\sigma)^5 = f(\sigma^5) = f(e) = e,$$

so the order of $f(\sigma)$ divides 5. Since 5 is prime, $f(\sigma) = e$ or $|f(\sigma)| = 5$. However, we claim that there are no elements of order 5 in S_4 , which means that $f(\sigma) = e$. To see that, note that by Lagrange's Theorem the order of any element in S_4 must divide $|S_4| = 4!$, but 5 does not divide $4!$.

We conclude that every 5-cycle in S_5 is in the kernel of f . Let's count the number of 5-cycles. There are $5!$ ways to order the numbers 1, 2, 3, 4, 5, but since permuting our list cyclicly will still correspond to the same cycle, there are $\frac{5!}{5} = 4!$ elements of S_5 that are 5-cycles. In particular, $|\ker f| \geq 4!$.

By the First Isomorphism Theorem,

$$S_5 / \ker f \cong \text{im } f.$$

Moreover, Lagrange's Theorem now tells us that

$$|\operatorname{im} f| = \frac{|S_5|}{|\ker f|} = \frac{5!}{|\ker f|} \leq \frac{5!}{4!} = 5.$$

In particular, it is not possible for f to be surjective. \square

Proof 2. Suppose that $f: S_5 \rightarrow S_4$ is a surjective group homomorphism. By the First Isomorphism Theorem,

$$S_4 \cong S_5 / \ker f.$$

Moreover, Lagrange's Theorem now tells us that

$$\frac{|S_5|}{|\ker f|} = |S_4| \implies |\ker f| = \frac{|S_5|}{|S_4|} = \frac{5!}{4!} = 5.$$

Thus $|\ker f| = 5$, and by 9(a), $\ker f$ must be cyclic. Consider any generator g for $\ker f$, which must then be an element of order 5.

Every element in S_5 can be written uniquely as a product of disjoint cycles, and its order is the least common multiple of the orders of each cycle. Moreover, any k -cycle has order k and the only element of order 1 is the identity. Since 5 is prime, g must be a product of 5-cycles; but we have only 5 elements to order, so we conclude that g must be a cycle of order 5.

Without loss of generality, assume $g = (12345)$; we can always rename the elements as such. So $\ker f = \langle (12345) \rangle$. Now the kernel of a homomorphism must be a normal subgroup, but we claim that $\langle (12345) \rangle$ is not a normal subgroup of S_5 . On the one hand,

$$\langle (12345) \rangle = \{e, \underset{g}{(12345)}, \underset{g^2}{(13524)}, \underset{g^3}{(15432)}, \underset{g^4}{(14253)}\}.$$

But if we take, for example, $\sigma = (12)$, then by a formula from a problem set we know that

$$\sigma(12345)\sigma^{-1} = (\sigma(1)\sigma(2)\sigma(3)\sigma(4)\sigma(5)) = (21345) \notin \langle (12345) \rangle.$$

We conclude that $\langle (12345) \rangle$ is not a normal subgroup of S_5 , and thus no such f can exist. \square