Extra Problems Solutions

Problem 1. Let $q(x) = x^4 - 2x^2 - 2 \in \mathbb{Q}[x]$.

a) Show that q is irreducible in $\mathbb{Q}[x]$.

b) The roots of q are

$$b_1 = \sqrt{1+\sqrt{3}}, \quad b_2 = \sqrt{1-\sqrt{3}}, \quad b_3 = -\sqrt{1+\sqrt{3}}, \quad \text{and } b_4 = -\sqrt{1-\sqrt{3}}.$$

Let $K_1 = \mathbb{Q}(b_1)$, $K_2 = \mathbb{Q}(b_2)$, and $F = \mathbb{Q}(\sqrt{3})$. Show that $K_1 \neq K_2$ and $K_1 \cap K_2 = F$.

c) Prove that K_1, K_2, and K_1K_2 are Galois over F.

d) Let $G = \mathrm{Gal}(K_1K_2/F)$. Show that G is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, and write out explicitly how this group acts on the roots of q.

e) Determine all of the subgroups $H \leq G$ and determine their corresponding fixed subfields $(K_1K_2)^H$.

f) Prove that the splitting field L of q over $\mathbb{Q}$ satisfies $[L : \mathbb{Q}] = 8$, and $\mathrm{Gal}(L/Q)$ is isomorphic to the dihedral group of order 8.

Hint: D_8 is the only non Hamiltonian group of order 8, meaning that D_8 is the only group of order 8 that has nonnormal subgroups.