Math 818 Lecture 24
March 24 — zoom

Announcements and reminders:

- Problem Set 6 due today
- Problem Set 7 posted
- Class on Monday back to normal (in person)
- On canvas, you can see your current grade in the class
- Coming up: anonymous feedback form. (canvas)
- Math 125: on Friday, April 28, class is CANCELED so you can attend the Math 125 events!

theorem (Jordan canonical form) F field $A \in M_n(F)$

Assume the characteristic polynomial c of A factors completely into linear factors there is an invertible matrix $P \in M_n(F)$ st

$$PAP^{-1} = \begin{bmatrix} J_{e_1}(r_1) & & \\ & \ddots & \\ & & J_{e_n}(r_n) \end{bmatrix} \sim A$$

similar to

each $r_i \in F$ is a root of c, $e_i \geq 1$

$(x-r_1)^{e_1}, \dots, (x-r_n)^{e_n}$ are the elementary divisors of V_t $(V = F^n)$ and the Jordan canonical form for A is unique up to order of the blocks

A is diagonalizable if there exists P st PAP^{-1} is diagonal

$V \xrightarrow{t} V$ is diagonalizable if $[t]_B^B$ is diagonal for some basis B

theorem $V \xrightarrow{t} V$ linear transformation $\dim_F(V) = n < \infty$

TFAE:

1. t is diagonalizable
2. t has a JCF, and it is diagonal
3. t has a JCF and the elementary divisors are all of the form $x - r$
4. Each invariant factor is a product of distinct linear forms
5. the minimal polynomial of t is a product of distinct linear forms.

Proof 1 $\Leftrightarrow$ 2 because JCF are unique

2 $\Leftrightarrow$ 3 by definition of JCF

having a JCF $\Longrightarrow$ invariant factors factor completely

elementary divisors come from invariant factors

3 $\Rightarrow$ 4 $\quad g_i = \underset{r_i \neq r_j}{(x-r_1)\cdots(x-r_n)} \Rightarrow$ elementary divisors $x - r_i$

$m = \text{lcm}(\text{inv factors}) = \text{lcm}(\text{elementary divisors}$

4 $\Rightarrow$ 5

④ $\Leftrightarrow$ ⑤ every invariant factor divides m

$$\underbrace{g_1 \mid \cdots \mid g_k}_{\text{inv factors}} = m$$

m distinct linear factors $\Rightarrow$ every g_i

<u>Remark</u> A and B matrices with JCFs

A and B are similar $\Leftrightarrow$ $JCF(A) = JCF(B)$
$(A = PBP^{-1})$

Moreover, if $A \overset{\text{similar}}{\sim} B$, then

A has a JCF $\Leftrightarrow$ B has a JCF.

New chapter:

Facts from 817:

Theorem 5.1 (Eisenstein's Criterion). *Suppose R is a domain and let $n \geqslant 1$, and consider the monic polynomial*

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in R[x].$$

If there exists a prime ideal P of R such that $a_0, \ldots, a_{n-1} \in P$ and $a_0 \notin P^2$, then f is irreducible in $R[x]$.

Theorem 5.2 (Gauss' Lemma). *Let R be a UFD with field of fractions F. Regard R as a subring of F and $R[x]$ as a subring of $F[x]$ via the induced map $R[x] \hookrightarrow F[x]$. If $f(x) \in R[x]$ is irreducible in $R[x]$, then $f(x)$ remains irreducible as an element of $F[x]$.*

Theorem 5.3. *Let R be a UFD with field of fractions F. Regard R as a subring of F and $R[x]$ as a subring of $F[x]$ via the induced map $R[x] \hookrightarrow F[x]$. If $f(x) \in R[x]$ is irreducible in $F[x]$ and the* gcd *of the coefficients of $f(x)$ is a unit, then $f(x)$ remains irreducible as an element of $R[x]$.*

subfield: F is a subfield of L

A field extension $F \subseteq L$ is an inclusion of fields F inside L.

Note: Given a field extension $F \subseteq L$, L is a vector space over F via

$$a \cdot v = \underbrace{av}_{\text{product in } L} \qquad a \in F, \quad v \in L$$

Notation: $F \subseteq L$ or L/F (L over F)

the degree of $F \subseteq L$ is $[L:F] := \dim_F(L)$.

A field extension is finite if the degree is finite.

Ex: $\mathbb{R} \subseteq \mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ basis $\{1, i\}$

$[\mathbb{C} : \mathbb{R}] = 2$.

Ex: $[\mathbb{R} : \mathbb{Q}] = \infty$

F field, $p \in F[x]$ irreducible, $\deg p \geq 2$
$\Longrightarrow L = F[x]/(p)$ is a field.

thm F field, $p \in F[x]$ irreducible $\deg p \geq 2$
$L = F[x]/(p)$

(1) $\begin{aligned} F &\longrightarrow L \\ a &\longmapsto a + (p) \end{aligned}$ is an inclusion of fields.

(2) $[L : F] = \deg(p)$

(3) $\bar{x} = x + (p) \in L$ is a root of $p \in L[x]$

Def $F \subseteq L$, $\alpha \in L$
$F(\alpha) :=$ smallest subfield of L containing α, F

$$F(\alpha) = \bigcap_{\substack{E \text{ field} \\ F \subseteq E \subseteq L \\ \alpha \in E}} E$$