Problem Set 13

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, our course notes, and the textbook.

I will post the .tex code for these problems for you to use if you wish to type your homework. If you prefer not to type, please *write neatly.* As a matter of good proof writing style, please use complete sentences and correct grammar. You may use any result stated or proven in class or in a homework problem, provided you reference it appropriately by either stating the result or stating its name (e.g. the definition of ring or Lagrange's Theorem). Do not refer to theorems by their number in the course notes or textbook.

Problem 1.

a) Show that the polynomial $x^4 + x + 1 \in \mathbb{Z}/2[x]$ is irreducible.

b) Give an explicit construction of a field with 16 elements.

Problem 2. Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$ is a Galois extension of degree 4 with Galois group that is a cyclic group of order 4.

Problem 3. Let L be the splitting field of $x^3 - 2$ over $\mathbb{Q}$.

a) Prove that there is a unique intermediate field K such that $[K : \mathbb{Q}] = 2$.

b) Find, with justification, a primitive element for K over $\mathbb{Q}$, that is, find an explicit α such that $K = \mathbb{Q}(\alpha)$.

Problem 4. Let $F \subseteq L$ be Galois field extension of degree 45. Prove there exists a unique intermediate field E such that $[E : F] = 5$.