Problem Set 6

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, our course notes, and the textbook.

I will post the .tex code for these problems for you to use if you wish to type your homework. If you prefer not to type, please *write neatly.* As a matter of good proof writing style, please use complete sentences and correct grammar. You may use any result stated or proven in class or in a homework problem, provided you reference it appropriately by either stating the result or stating its name (e.g. the definition of ring or Lagrange's Theorem). Do not refer to theorems by their number in the course notes or textbook.

Problem 1. Let F be a field and consider a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ in $F[x]$ with $n \geqslant 1$.

a) Show that the principal ideal $(f(x))$ is a subspace of the F-vector space $F[x]$.

b) Show that the set $B = \{\overline{1}, \overline{x}, \ldots, \overline{x^{n-1}}\}$, where $\overline{x^i} = x^i + (f(x))$, is a basis for the quotient F-vector space $F[x]/(f(x))$.

c) Consider the linear transformation $l_x : F[x]/(f(x)) \to F[x]/(f(x))$ defined by $l_x(v) = \overline{x}v$ for any $v \in F[x]/(f(x))$. Find the matrix representing l_x in the basis B from part b).

Problem 2. Let $V = \mathbb{R}^3$ with the standard basis $B = \{e_1, e_2, e_3\}$ and let $t : V \to V$ be the linear transformation represented by the matrix

$$[t]_B^B = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

a) Find the invariant factor decomposition of the $\mathbb{R}[x]$-module V_t.

b) Find the characteristic and minimal polynomials of t.

c) Find the rational canonical form of t.

d) Find the Jordan canonical form of t.

Problem 3. Let F be a field, let V and W be vector spaces over F, let $a\colon V \to V$ and $b\colon W \to W$ be linear transformations and let V_a and W_b be the $F[x]$-modules they determine.

a) Show that a function $g : V_a \to W_b$ is an $F[x]$-module homomorphism if and only if

   (1) $g : V \to W$ is a linear transformation and

   (2) $g \circ a = b \circ g$.

b) Suppose that $V = F^m = W$, and let $A, B \in \mathrm{M}_m(F)$ be the matrices representing the linear transformations a and b, respectively, in the standard basis of F^m. Show that there is an $F[x]$-module isomorphism $V_a \cong W_b$ if and only if the matrices A and B are similar.

Problem 4. Let F be a field and n a positive integer. We say an $n \times n$ matrix A with entries in F is **unipotent** if $A - I_n$ is nilpotent, meaning that $(A - I_n)^k = 0$ for some $k \geqslant 1$. For the field $F = \mathbb{Q}$, find (with complete justification) the number of similarity classes of 4×4 unipotent matrices and give an explicit representative for each class.