Problem Set 6

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, our course notes, and the textbook.

I will post the .tex code for these problems for you to use if you wish to type your homework. If you prefer not to type, please *write neatly.* As a matter of good proof writing style, please use complete sentences and correct grammar. You may use any result stated or proven in class or in a homework problem, provided you reference it appropriately by either stating the result or stating its name (e.g. the definition of ring or Lagrange's Theorem). Do not refer to theorems by their number in the course notes or textbook.

Problem 1. Determine, with justification, if the following two matrices with complex entries are similar.

$$A = \begin{bmatrix} 0 & -4 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Proof. Two matrices are similar if and only if they have the same Rational Canonical Form, and equivalently they have the same invariant factors. Notice that the matrix A is already in Rational Canonical form:

$$A = C(4 - 4x + x^2) \oplus C(4 - 4x + x^2).$$

Note that $4 - 4x + x^2 = (x - 2)^2$. In particular, the invariant factors of A are $(x - 2)^2, (x - 2)^2$.

As for B, we have

$$xI_4 - B = \begin{bmatrix} x-2 & 0 & 0 & 0 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & x-2 & 0 \\ 0 & 0 & -1 & x-2 \end{bmatrix} \xrightarrow{R3 \leftrightarrow R4} \begin{bmatrix} x-2 & 0 & 0 & 0 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & -1 & x-2 \\ 0 & 0 & x-2 & 0 \end{bmatrix}$$

$$\xrightarrow{C4 \to C4 + (x-2)C3} \begin{bmatrix} x-2 & 0 & 0 & 0 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & x-2 & (x-2)^2 \end{bmatrix} \xrightarrow{R4 \to C4 + (x-2)R3} \begin{bmatrix} x-2 & 0 & 0 & 0 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & (x-2)^2 \end{bmatrix}$$

After multiplying the third row by -1 ad switching rows and columns, we conclude that $xI_4 - B$ has Smith Normal Form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & x-2 & 0 \\ 0 & 0 & 0 & x-2 \end{bmatrix}.$$

In particular, the invariant factors of B are $x-2, x-2, (x-2)^2$, so B and A are not similar. In fact, the Rational Canonical Form of B is

$$RCF(B) = C(x-2) \oplus C(x-2) \oplus C((x-2)^2) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 1 & 4 \end{bmatrix}. \quad \square$$

Problem 2. Let F be a field.

a) Let A and B be two 3×3 matrices with entries in F. Prove A and B are similar if and only if they have the same characteristic polynomial and the same minimum polynomial.

Proof. For a 3×3 matrix A, the characteristic polynomial c_A has degree 3, and the minimal polynomial m_A has degree 1, 2, or 3, since it divides c_A and has degree at least 1. Recall that c_A is the product of all the invariant factors of A, while m_A is the largest invariant factor (meaning, the one that is divisible by all the others). Recall also that two matrices are similar if and only if they have the same invariant factors if and only if they have the same RCF.

Suppose that A and B are similar. Since the minimal and the characteristic polynomials are determined by the RCF, two similar matrices must have the same minimal polynomial and the same characteristic polynomial.

Suppose $m_A = m_B$ and $c_A = c_B$. If $\deg(m_A) = \deg(m_B) = 3$ then it follows that m_A and m_B are the only invariant factors for A and B respectively, and so $A \sim C(m_A) = C(m_B) \sim B$. If $\deg(m_A) = \deg(m_B) = 2$ then it follows that A and B each have exactly two invariant factors (since their degrees have to add up to 3) and they are $\frac{c_A}{m_A}, m_A$ and $\frac{c_B}{m_B}, m_B$. In particular, the invariant factors for A and B are the same and so

$$A \sim C(m_A) \oplus C(c_A/m_A) = C(m_B) \oplus C(c_B/m_B) \sim B.$$

If $\deg(m_A) = \deg(m_B) = 1$, then all the other invariant factors of both A and B must have degree 1, since they all divide the minimal polynomial. Therefore, A and B each have exactly three invariant factors m_A, m_A, m_A and m_B, m_B, m_B. Therefore,

$$A \sim C(m_A) \oplus C(m_A) \oplus C(m_A) = C(m_B) \oplus C(m_B) \oplus C(m_B) \sim B. \quad \square$$

b) Show, by way of an example with justification, that the statement in part a) would become false if 3×3 were replaced by 4×4.

Solution. Let F be a field and let

$$A = C((x-1)^2) \oplus C((x-1)^2) \quad \text{and} \quad B = C(x-1) \oplus C(x-1) \oplus C((x-1)^2).$$

Then both A and B are in rational canonical form but $A \neq B$, and so A and B are not similar. However, $c_A = (x-1)^4 = c_B$ and $m_A = (x-1)^2 = m_B$. Therefore A and B have the same minimal polynomial and the same characteristic polynomial, but are not similar.

Problem 3. Let F be any field. Up to similarity, how many matrices in $\mathrm{M}_5(F)$ of the form

$$A = \begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

are there? Justify.

Proof. First, note that since all such matrices are upper triangular, the characteristic polynomial is easy to compute, as it is given by the product of the diagonal entries of $xI_5 - A$: all such matrices have characteristic polynomial $(x-1)^5$. In particular, they all have a JCF. The possible JCFs are

$$J_{a_1}(1) \oplus \cdots \oplus J_{a_s}(1)$$

with $a_1 + \cdots + a_s = 5$ and $a_i \geqslant 1$. But order does not matter. So, the number of nonsimilar JCFs of this form is equal to the number of ways are writing 5 as $a_1 + \cdots + a_s = 5$ with $1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_s$ for $s \geqslant 1$. There are 7 such ways: $5 = 5, 1 + 4 = 5$, $2 + 3 = 5$, $1 + 2 + 2 = 5$, $1 + 1 + 3 = 5$, $1 + 1 + 1 + 2 = 5$, $1 + 1 + 1 + 1 + 1 + 1 = 5$. Thus there are 7 similarity classes of matrices of this form. □

Problem 4. Consider the vector space over $\mathbb{C}$ given by

$$V = \{p \in \mathbb{C}[x] \mid \deg(p) < n\}.$$

Consider the linear transformations $V \to V$ given as follows:

$$T(p) = p' \text{ and } D(p) = xp'.$$

Determine the Jordan Canonical Form of T and D.

Proof. Note that $T^{\circ n}$, meaning T composed with itself n times, is the 0 operator, but $T^{\circ n-1} \neq 0$ since $T^{\circ n-1}(x^{n-1}) = (n-1)!$. This proves that the minimum polynomial of D is $\mp_D(t) = t^n$. Note: I'll use t for the variable of my invariant factors and elementary divisors to avoid conflict with the elements of V. Since the minimum polynomial has degree equal to the dimension of V, it is equal to the characteristic polynomial, and there is just one invariant factor: t^n. It follows that the JCF is $J_n(0)$:

$$JCF(T) = J_n(0) = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix}.$$

As for D, for the basis $B = \{1, x, \ldots, x^{n-1}\}$, the matrix $[D]_B^B$ representing D is diagonal with diagonal entries $0, 1, 2, \ldots, n-1$. This matrix is already in JCF. □

Problem 5. If $F \subseteq L$ is a field extension and $\alpha \in L$, show, using only the definition of $F(\alpha)$, that

$$F(\alpha) = \left\{ \frac{g(\alpha)}{f(\alpha)} \mid g(x), f(x) \in F[x], f(\alpha) \neq 0 \right\}.$$

Proof. Recall the definition for $F(\alpha)$:

$$F(\alpha) = \bigcap_{\substack{E \text{ field} \\ F \cup \{\alpha\} \subseteq E \subseteq L}} E$$

and let

$$S := \left\{ \frac{g(\alpha)}{f(\alpha)} \mid g(x), f(x) \in F[x], f(\alpha) \neq 0 \right\}.$$

We have several things to show:

- S is a field contained in L and containing F and α:

  By definition, S is the fraction field of the domain $F[\alpha] = \{f(\alpha) \mid f \in F[x]\}$. Since $F[\alpha] \subseteq L$, we have by closure of L under multiplication and taking inverses that $S = \mathrm{Frac}(F[\alpha]) \subseteq L$. Furthermore, note that
  $$F \cup \{\alpha\} \subseteq F[\alpha] \subseteq S.$$

- $F(\alpha) \subseteq S$:

  By definition, $F(\alpha)$ is contained in all fields E such that $F \cup \{\alpha\} \subseteq E \subseteq L$, and S is one of the fields by the previous item.

- $S \subseteq F(\alpha)$:

  Note that since the intersection of an arbitrary collection of fields is a field, $F(\alpha)$ is a field. The definition guarantees that $F \cup \{\alpha\} \subseteq F(\alpha)$. By closure of $F(\alpha)$ under addition and multiplication starting with elements of $F \cup \{\alpha\}$ it follows that $f(\alpha) \in F(\alpha)$ for any polynomial $f \in F[x]$. Furthermore by closure of $F(\alpha)$ under inverses and multiplication we deduce that $\frac{g(\alpha)}{f(\alpha)} \in F(\alpha)$ for all $f, g \in F[x]$ such that $f(\alpha) \neq 0$, thus $S \subseteq F(\alpha)$ as desired.

The second and third items now establish that $F(\alpha) = S$. $\square$