Midterm

Instructions: You may use any results proved in class or in the problem sets, except for the specific question being asked. You should clearly state any facts you are using. Remember to show all your work, and to write clearly and using complete sentences. No calculators, notes, cellphones, smartwatches, or other outside assistance allowed.

Quick questions

Problem 1. Let R be a ring, M be an R-module, and N be a submodule of M. Describe the submodules of M/N according to the Lattice Isomorphism Theorem.

Problem 2. What does the Classification of finitely generated modules over a PID say? State either one of the two theorems we gave this name to.

Problem 3. For each of the following, give an example or briefly explain why one doesn't exist:

a) A ring R and an R-module M such that $\operatorname{ann}(M) = 0$ but M is not free (both R and M).

b) A 3×3 matrix A with entries in $\mathbb{Z}$ that presents a 2-generated $\mathbb{Z}$-module M (both A and M).

Problem set questions

Choose **1** of the following problems.

Problem 4. Show that a left R-module M is cyclic if and only if $M \cong R/I$ for some left ideal I.

Problem 5. Let R be a commutative ring with $1 \neq 0$. Show that if every R-module is free then R is a field.

Problem 6. Show that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

Old qualifying exam questions

Choose **1** of the following problems.

Problem 7. Let R be a domain. We say that a subset S of an R-module M is **maximally linearly independent** if it is linearly independent and every subset T of M properly containing S is not linearly independent. Recall that we say a module M is **torsion** if for every $m \in M$ there exists a nonzero $r \in R$ such that $rm = 0$.

a) Let S be a linearly independent set of M and let N be the submodule generated by S. Show that S is maximally linearly independent if and only if M/N is torsion.

b) Suppose that for every R-module M, every maximally linearly independent set of M generates M. Show that R must be a field.

Problem 8. Let R be a commutative ring with $1 \neq 0$. Let $f\colon R^a \to R^b$ be a surjective R-module homomorphism. Show that $a \geqslant b$.

New problems

Choose any **2** of the following problems.

Problem 9. Let R be a commutative ring and let I and J be ideals of R. Show that if $R/I \cong R/J$ then $I = J$.

Problem 10. Let V be a finite dimensional vector space over a field F and let $t : V \to V$ be a linear transformation. Prove that the following are equivalent:

(1) t is injective,

(2) t is surjective,

(3) for any basis B of V, $t(B)$ is a basis of V.

Problem 11. Suppose M is an abelian group (that is, a $\mathbb{Z}$-module) such that $|M| = 400$ and $\operatorname{ann}_{\mathbb{Z}}(M) = (20)$. Determine all the possibilities for M, up to isomorphism.