

Problem Set 1

Problem 1.

- a) Install Macaulay2.¹ Hardcore version: install emacs and run Macaulay2 through emacs.
- b) Make an .m2 file setting up a polynomial ring R over a field k , a nontrivial ideal I in R , the R -module $M = R/I$ and the ring $S = R/I$.

Problem 2.

 Use Macaulay2 to find:

- a) A presentation for the \mathbb{Q} -algebra $\mathbb{Q}[xy, xu, yv, uv] \subseteq \mathbb{Q}[x, y, u, v]$.
- b) A presentation for the k -algebra U , where $k = \mathbb{Z}/101$ and

$$k \begin{bmatrix} ux & uy & uz \\ vx & vy & vz \end{bmatrix} \subseteq \frac{k[u, v, x, y, z]}{(x^3 + y^3 + z^3)}.$$

Problem 3. Let k be a field, and $R := k[x^2, x^3] \subseteq S := k[x]$. Let $I = x^2R$ be the ideal generated by x^2 in R . Show that $IS \cap R \supsetneq I$, where IS is the smallest ideal in S that contains I :

$$IS = \left\{ \sum_{i=1}^r a_i s_i \mid a_i \in I \text{ and } s_i \in S \right\}.$$

Problem 4. Let k be a field. Is the extension $k[x^{42}, y^{17} + y^7, z^{73} + z] \subseteq k[x, y, z]$ module-finite?

Problem 5. Find $\overline{\mathbb{Z}}$, the integral closure of \mathbb{Z} in its field of fractions \mathbb{Q} .

Problem 6. If $R[x]$ is Noetherian, must R be Noetherian?

Problem 7. Suppose that every ascending chain of prime ideals in R stabilizes. Must R be a noetherian ring? Spoiler alert: there is a hint in the footnotes.²

Problem 8. Show that R is a Noetherian ring if and only if every prime ideal of R is finitely generated. Spoiler alert: there's a hint in the footnotes.³

¹If you don't have access to a computer, or if your computer runs only Windows, come talk to me about it.

²Hint: it may be helpful to think of how we could construct rings with very few prime ideals.

³Use Zorn's Lemma to show that if R is a ring that is not noetherian, the set of ideals that are not finitely generated has a maximal element; show that element is a prime ideal.