Problem Set 2

Problem 1. Given two ideals I and J, the *product* of I and J is the ideal

$$IJ := (fg \mid f \in I \text{ and } g \in J).$$

a) Show that $IJ \subseteq I \cap J$.

b) If I and J satisfy $I + J = R$, then $IJ = I \cap J$.

c) In general, $IJ \neq I \cap J$. Find an example of a ring R and ideals I and J with $IJ \neq I \cap J$.

Problem 2. A prime ideal P in a ring R is a *minimal prime* if for every prime ideal Q in R,

$$Q \subseteq P \implies Q = P.$$

Show that every prime ideal in a ring R contains some minimal prime.

Let R be a ring, and M an R-module. The *Nagata idealization* of (R, M) is the ring $R \ltimes M$ such that

- as a set, $R \ltimes M = R \times M$;
- the addition is $(r, m) + (s, n) = (r + s, m + n)$;
- the multplication is $(r, m)(s, n) = (rs, sm + rn)$.

Then $R \ltimes M$ with the operations specified about is a ring.

Problem 3. Consider an extension of rings $A \subseteq B \subseteq C$. In this problem, we will construct an example of such an extension such that $A \subseteq C$ is module-finite, but $A \subseteq B$ is not.[1]

a) Can you find such an extension with A Noetherian?

b) Let R be a ring that is not Noetherian, and I an ideal that is not finitely generated. Show that[2] $R \subseteq R \ltimes I \subseteq R \ltimes R$, that $R \subseteq R \ltimes R$ is module-finite, but $R \subseteq R \ltimes I$ is not.

Problem 4.

a) In Macaulay2, set up $A = \mathbb{Q}[s^2, st, t^2]$ as a $\mathbb{N}^2$-graded ring with the grading induced by setting s^2, st, t^2 as homogeneous elements of degrees

$$\deg(s^2) = (2, 0) \quad \deg(st) = (1, 1) \quad \deg(t^2) = (0, 2).$$

b) The ring $R = k[t^3, t^{13}, t^{42}]$ is a graded subring of $k[t]$ with the standard grading, meaning that the graded structure on $k[t]$ induces a grading on R. Set up R (with this grading) in Macaulay2.

[1] We remarked before that such examples exist, but we didn't construct one.

[2] Note that R is a subring of $R \ltimes M$ (via the inclusion $r \mapsto (r, 0)$), and as an R-module, $R \ltimes M \cong R \oplus M$.

Problem 5. The curve C parametrized by

$$\{(t^3, t^4, t^5) : t \in \mathbb{Q}\}$$

in $\mathbb{A}^3_{\mathbb{Q}}$ is a variety. Use Macaulay2 to find $\mathcal{I}(C) \subseteq \mathbb{Q}[x, y, z]$. Is C irreducible?

Problem 6. Let X be the solution set of the system of equations

$$\begin{cases} y^4 - 2\,x\,y^2 z + x^2 z^2 = 0 \\ x^4 y^3 - x^5 y\,z - y^2 z^3 + xz^4 = 0 \\ x^5 y^2 - x^6 z - y^3 z^2 + xyz^3 = 0 \\ x^9 + x^3 y^3 z - 3x^4 y z^2 + z^5 = 0 \end{cases}$$

over $\mathbb{Z}/73$. Find $\mathcal{I}(X)$.

Problem 7. Show that the functions $\mathcal{Z}$ and $\mathcal{I}$ have the following properties:

a) If $I = (T)$ is the ideal generated by the elements of T, then $\mathcal{Z}(T) = \mathcal{Z}(I)$.

b) For any field k, we have $\mathcal{Z}(0) = \mathbb{A}^n_k$ and $\mathcal{Z}(1) = \emptyset$.

c) $\mathcal{I}(\emptyset) = (1) = K[x_1, \ldots, x_n]$ (the improper ideal).

d) $\mathcal{I}(\mathbb{A}^n_k) = (0)$ if and only if k is infinite.

e) If $I \subseteq J \subseteq K[x_1, \ldots, x_n]$ then $\mathcal{Z}(I) \supseteq \mathcal{Z}(J)$.

f) If $S \subseteq T$ are subsets of $\mathbb{A}^n_k$ then $\mathcal{I}(S) \supseteq \mathcal{I}(T)$.

Problem 8. In this problem, we will show that the union and intersection of varieties is a variety.

a) Given two ideals I and J in $k[x_1, \ldots, x_d]$, $\mathcal{Z}(I) \cap \mathcal{Z}(J) = \mathcal{Z}(I + J)$.

b) Given two ideals I and J in $k[x_1, \ldots, x_d]$, $\mathcal{Z}(I) \cup \mathcal{Z}(J) = \mathcal{Z}(IJ) = \mathcal{Z}(I \cap J)$.