Problem Set 3

Problem 1. Show that

$$V(I) = \bigcup_{P \in \operatorname{Min}(I)} V(P)$$

and conclude that

$$\sqrt{I} = \bigcap_{P \in \operatorname{Min}(I)} P.$$

Problem 2. Let I and J be ideals in a ring R.

a) Show that $\sqrt{\sqrt{I}} = \sqrt{I}$.

b) Show that if $I \subseteq J$, then $\sqrt{I} \subseteq \sqrt{J}$.

c) Show that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

d) Show that $\sqrt{I^n} = \sqrt{I}$ for all $n \geqslant 1$.

e) Show that if P is a prime ideal, then $\sqrt{P^n} = P$ for all $n \geqslant 1$.

Problem 3.

a) Show that P is a prime ideal if and only if P satisfies the following property: given ideals I and J in R, if $IJ \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.

b) Show that if P is a prime ideal and $P = I \cap J$ for some ideals I and J, then $P = I$ or $P = J$.

Problem 4. Let R be a ring of characteristic $p > 0$. The *Frobenius map* on R is the map

$$\begin{aligned} R &\xrightarrow{F} R \\ r &\longmapsto r^p \end{aligned}$$

a) Show that the Frobenius map is a ring homomorphism.

b) Show that the Frobenius map is module-finite if and only if it is algebra-finite.

c) Show that the map on spectra induced by the Frobenius map is the identity map.

Problem 5. Given any subset $X \subseteq \mathbb{A}^d_{\mathbb{C}}$, show that $\mathcal{Z}(\mathcal{I}(X))$ is the closure of X in the Zariski topology.

Problem 6. Consider the inclusion map $k[xy, xz, yz] \subseteq k[x, y, z]$. Is the induced map on Spec surjective? If not, give an explicit prime not in the image.

Problem 7. Let $R = \mathbb{Q}[x, y, z]/(xy, yz)$, and let $I = (xz)$ in R. The R-module I/I^2 is cyclic; write it explicitly as a quotient of R.

Problem 8. Let $R = \mathbb{R}[x, y]/(xy)$ and consider the ideal $P = (x - 1)$ in R. Show that P is a maximal ideal in R, and that $R_P \cong \mathbb{R}[t]_{(t-1)}$. What does this mean geometrically?