Problem Set 4

Problem 1. Let M a Noetherian R-module.

a) Show that every surjective R-module homomorphism $M \longrightarrow M$ is an isomorphism.

b) Must an injective R-module homomorphism $M \longrightarrow M$ be an isomorphism?

Problem 2. Show that M is a Noetherian R-module if and only if M is finitely generated and $R/\operatorname{ann} M$ is a Noetherian ring.

Problem 3.

a) Let $(R, \mathfrak{m})$ be a local domain that is not a field. Show that the fraction field of R is not a finitely generated R-module.

b) Let I be a proper ideal in a local ring $(R, \mathfrak{m})$, and M a finitely generated R-module. Show that if $IM = M$, then $M = 0$.

Problem 4. Is there an upper bound for the minimal number of generators of an ideal in $k[x, y]$? Here k is a field. Either prove there is an upper bound or find ideals I with arbitrarily large $\mu(I)$.

Problem 5. Show that the containment of ideals is a local property: given any ideals I and J in a ring R,

$$I \subseteq J \iff I_P \subseteq J_P \text{ for all } P \in \operatorname{Spec}(R).$$

Problem 6.

a) Describe $\operatorname{supp}(I/I^2)$, where $I = (xz)$ in $R = \mathbb{C}[x, y, z]/(xy, yz)$.

b) Find all the minimal primes of $J = (ab, bc, cd, ad)$ in $k[a, b, c, d]$ over any field k.

c) Find the minimal primes of the ring S, where

$$S = \mathbb{Q}\begin{bmatrix} ux & uy & uz \\ vx & vy & vz \end{bmatrix} \subseteq \frac{\mathbb{Q}[u, v, x, y, z]}{(x^3 + y^3 + z^3)}.$$

d) Let I be the defining ideal of the curve parametrized by (t^{13}, t^{42}, t^{73}) over $\mathbb{Q}$. Find $\mu(I)$ and a minimal generating set for I.

Problem 7. Let $R = \mathbb{Q}[x, y, z]$, and $I = (x^3, x^2y, x^2z, xyz)$. Find a prime filtration of R/I and a primary decomposition of I.

Problem 8. Let R be a Noetherian ring, $x \in R$, and I be an ideal. Show that if x is not in any of the associated primes of R/I, then $(x)I = (x) \cap I$.