Problem Set 5

Problem 1. Let $R = \mathbb{Z}[\sqrt{-5}]$. While $6 \in R$ cannot be written as a unique product of irreducibles, we are going to show that the ideal $I = (6)$ does have a unique primary decomposition. Unfortunately, Macaulay2 cannot take primary decompositions over $\mathbb{Z}$, but this one we can do the old fashioned way.

a) Prove that (2) is a primary ideal.

b) Prove that (3) is *not* a primary ideal.

c) Prove that $(3, 1 + \sqrt{-5})$ and $(3, 1 - \sqrt{-5})$ are both primary.

d) Show that $(6) = (2) \cap (3, 1 + \sqrt{-5}) \cap (3, 1 - \sqrt{-5})$.

e) Why is this primary decomposition unique?

Problem 2. Let R be a Noetherian ring. Let I be an ideal in R, and $x \in R$. The *saturation* of I with respect to x is the ideal

$$(I : x^\infty) := \bigcup_{n=1}^{\infty} (I : x^n).$$

a) Let Q be a P-primary ideal. Show that

$$(Q : x^\infty) = \begin{cases} Q & \text{if } x \notin P \\ R & \text{if } x \in P \end{cases}.$$

b) Show that $(I : x^\infty) = (I : x^n)$ for some n.

c) Show that $(I \cap J : x^\infty) = (I : x^\infty) \cap (J : x^\infty)$ for any ideals I and J.

d) Let $I = Q_1 \cap \cdots \cap Q_k$ be a primary decomposition, and $x \in R$. Show that

$$(I : x^\infty) = \bigcap_{x \notin \sqrt{Q_i}} Q_i.$$

Problem 3. Let I be a radical ideal in a Noetherian ring R. A beautiful theorem of Brodmann says that the set

$$\bigcup_{n \geqslant 1} \operatorname{Ass}(R/I^n)$$

is finite. Show that there exists an element x such that:

- x is contained in every embedded prime of I^n for every n, and
- $x \notin P$ for all $P \in \operatorname{Min}(I)$.

Conclude that there exists $x \in R$ such that $I^{(n)} = (I^n : x^\infty)$ for all $n \geqslant 1$.

Problem 4. Consider s points $P_1 = (a_{11}, \ldots, a_{1d}), \ldots, P_s = (a_{s1}, \ldots, a_{sd})$ in $\mathbb{A}^d$, and let I be the corresponding radical ideal in $\mathbb{C}[x_1, \ldots, x_d]$. Show that for all $n \geqslant 1$,

$$I^{(n)} = \bigcap_{i=1}^{s} (x_1 - a_{i1}, \ldots, x_d - a_{id})^n.$$

Problem 5. Let R be a Noetherian ring.

a) Show that if $\mathfrak{m}$ is any maximal ideal in R, then $\mathfrak{m}^n$ is $\mathfrak{m}$-primary for any $n \geqslant 1$.

b) If R is a domain, then

$$\bigcap_{n \geqslant 1} I^n = 0.$$

for any proper ideal I in R.[1]

Problem 6. If $(R, \mathfrak{m})$ is a Noetherian local ring, show that M has finite length if and only if M is finitely generated and $\mathfrak{m}^n M = 0$ for some n.

Problem 7. Let k be a field, and $R = k[a,b,c,d]/(ad - bc)$. Find prime ideals P and Q in R such that $\operatorname{ht}(P) + \operatorname{ht}(Q) < \operatorname{ht}(P + Q)$.

Problem 8.

a) Find the height of $J = (ab, bc, cd, ad)$ in $k[a,b,c,d]$ over any field k, and the dimension of $k[a,b,c,d]/J$.

b) Find the dimension of the ring S, where $S = \mathbb{Q}[x^3y^3, x^3y^2z, x^2z^3] \subseteq \mathbb{Q}[x,y,z]$.

c) Let I be the defining ideal of the curve parametrized by (t^{13}, t^{42}, t^{73}) over $\mathbb{Q}$. Find the height of I, and notice that $\operatorname{height}(I) < \mu(I)$.

d) Let $R = \mathbb{Q}[x,y,z]$, and $I = (x^3, x^2y, x^2z, xyz)$. Find the dimension of R/I and the height of I.

e) Find the dimension of the module I/I^2, where $I = (xz)$ in $R = \mathbb{C}[x,y,z]/(xy, yz)$.

[1] Hint: first, do the case where I is a maximal ideal in R. Be wise, localize!