Sidenote about colons and annihilators

Def: $\quad \operatorname{ann}(M) = \{x \in R \mid xm = 0 \text{ for all } m \in M\}$

$$(I :_R J) = \{x \in R \mid xJ \subseteq I\}$$

Exercise $\operatorname{ann}(M)$, $(I :_R J)$ are ideals in R and

$$\operatorname{ann}(M) = (0 :_R M)$$

Remarks:

1) If $M = R \cdot m$ is a one-generated R-mod, then

$$M \cong R/I \text{ for some ideal } I \subseteq R$$

Also,

$$\begin{cases} I \cdot (R/I) = 0 & \Longrightarrow \operatorname{ann}(R/I) = I \\ g \cdot (R/I) = 0 \Longrightarrow g \in I & \end{cases}$$

so $\quad M \cong R/\operatorname{ann}(M) \quad$ if M is <u>one</u> generated.

2) Any R-module M is naturally an R/I-module with the <u>same</u> structure it has as an R-mod if $I \subseteq \operatorname{ann}(M)$: $\quad (x+I) \cdot m = x \cdot m$

3) $\operatorname{ann}(M/N) = N :_R M$

Local Rings

A ring R is local if it has only one maximal ideal.

$$\Updownarrow$$

$\{a \in R : a \text{ is not a unit}\}$ is an ideal

Notation $(R, m) :=$ local ring with max ideal m $\qquad (R, m, k)$, k = residue field R/m

Note For some authors, local = local Noetherian. Not for us.

Examples

1) $\mathbb{Z}/(p^n)$ is local with maximal ideal (p)

2) $\mathbb{Z}_{(p)} := \left\{ \frac{a}{b} \in \mathbb{Q} \mid p \nmid b \text{ when } (a,b)=1 \right\}$

3) $k[[x]]$ is local!

$$\sum_{n=0}^{\infty} a_n x^n \text{ is invertible} \overset{\text{exercise}}{\Longleftrightarrow} a_0 \neq 0$$

nonunits $= (x) =$ unique maximal ideal

4) $k[[x_1, \ldots, x_d]]$ is local with maximal ideal $(x_1, \ldots, x_d)$

5) $R = k[x_1, \ldots, x_d]$ is <u>NOT</u> local.

Def char R : integer $n \geq 0$

$$(n) = \ker \begin{pmatrix} \mathbb{Z} \longrightarrow R \\ a \longmapsto a \cdot 1_R \end{pmatrix}$$

so the smallest n st $\underbrace{1 + \cdots + 1}_{n \text{ times}} = 0$, 0 if no such n exists.

Prop (R, m, k) local ring. One of the following holds:

1. char R = char $k = 0$ R has equal characteristic 0
2. char $R = 0$, char $k = p$ R has mixed characteristic $(p, 0)$
3. char R = char $k = p$ prime R has characteristic p
4. char $R = p^n$, char $k = p$ prime

If R is reduced, 1, 2, or 3 holds.

Proof See notes.

Note $R = \bigoplus_{n \geq 0} R_n$, $R_0 = k$ a field $m = \bigoplus_{n \geq 1} R_n$

(R, m, k) behaves a lot like a local ring

statements about ideals $\rightsquigarrow$ homogeneous ideals
modules $\rightsquigarrow$ graded modules

Localization R ring

$\emptyset \neq W$ multiplicative set ($1 \in W$, $a, b \in W \Rightarrow ab \in W$)

the localization of R at W is the ring

$$W^{-1}R := \left\{ \frac{r}{w} \;\middle|\; r \in R, w \in W \right\} / \sim$$

$$\frac{r}{w} \sim \frac{r'}{w'} \iff u(rw' - r'w) = 0 \text{ for some } u \in W$$

the operations on $W^{-1}R$ are given by

$$\frac{r}{w} + \frac{s}{v} := \frac{rv + sw}{wv} \qquad \frac{r}{w} \cdot \frac{s}{v} := \frac{rs}{wv}$$

zero: $\frac{0}{1}$ identity: $\frac{1}{1}$

canonical map
$$R \longrightarrow W^{-1}R$$
$$r \longmapsto \frac{r}{1}$$

Remark R domain, $\frac{r}{w} \sim \frac{r'}{w'} \iff rw' = r'w$

$$\Rightarrow \quad R \subseteq W^{-1}R \subseteq \operatorname{Frac}(R) = (R \setminus \{0\})^{-1}R$$

Universal property

R ring

$0 \notin W$ multiplicative set

S R-algebra where every $w \in W$ is a unit

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & W^{-1}R \arrow[dl, dashed, "\exists!"] \\ S & \end{tikzcd}$$

$\sim$ $W^{-1}R$ is the smallest R-algebra st every element in W is a unit

Most important Examples

1) $f \in R$ $\qquad R_f := W^{-1}R$ for $W = \{1, f, f^2, \dots\}$

2) Localization at a prime

p prime in R $\Rightarrow$ $R \setminus p$ is a multiplicative set

$R_p := (R \setminus p)^{-1} R$ $\qquad$ R localized at p

In fact, $(R_p, p_p, R/p)$ is a local ring

3) $W =$ nonzero divisors of R

$W^{-1}R$ total ring of fractions of R

R domain $\Rightarrow$ $\operatorname{frac}(R) = W^{-1}R = R_{(0)}$

Examples

1) $k[x_1,\dots,x_d]_{(x_1,\dots,x_d)}$ = ring of rational functions with nonzero constant term in denominator

2) If $k=\bar{k}$, $I=\sqrt{I}$ then

$$\left(k[x_1,\dots,x_d]/I\right)_{(x_1,\dots,x_d)} = k[X]_{m_{\underline{0}}}$$ for some affine variety X

where $m_{\underline{0}}$:= maximal ideal corresponding to $\underline{0}\in X\subseteq \mathbb{A}^d$

$\hookrightarrow$ this is the local ring of $\underline{0}\in X$

Radical ideals in this ring $\equiv$ subvarieties of X containing $\underline{0}$