Last time Localization of a ring R at a multiplicative set W

$$W^{-1}R := \left\{ \frac{x}{w} \mid x \in R,\ w \in W \right\} / \sim \qquad \text{where}$$

$$\frac{x}{w} \sim \frac{x'}{w'} \iff u(xw' - x'w) = 0 \text{ for some } u \in W$$

canonical map: $\begin{array}{ccc} R & \longrightarrow & W^{-1}R \\ x & \longmapsto & \frac{x}{1} \end{array}$ (not necessarily injective)

eg, $R = \dfrac{k[x,y]}{(xy)}$ In $R_{(x)}$, $\dfrac{x}{1} = \dfrac{yx}{y} = \dfrac{0}{y} = \dfrac{0}{1}$
$y \notin (x)$

Localization at a prime R ring, P prime

Localization of R at P is a local ring $(W = R \setminus P)$
(R_P, P_P)

with residue field $k(p) = R_P / P_P \cong (R/P)_P$

will show:

Ideals in R_P		Ideals in R
I_P	$\longleftrightarrow$	$I \subseteq P$
primes Q_P	$\longleftrightarrow$	primes $Q \subseteq P$

Many properties of rings/ideals/modules are local, meaning they can be checked at all prime ideals (by localizing)

Eg, containments are local: $I \subseteq J \iff I_P \subseteq J_P \ \forall P \in \operatorname{Spec} R$

Def M R-mod

$W \subseteq R$ multiplicative set

$W^{-1}M := \left\{ \frac{m}{w} \mid m \in M,\ w \in W \right\} / \sim$

$\frac{m}{w} \sim \frac{m'}{w'}$ if $u(mw' - m'w) = 0$ for some $u \in W$

this is an R_W-module via

$$\frac{m}{w} + \frac{m'}{w'} = \frac{mw' + m'w}{ww'} \qquad \frac{r}{w} \cdot \frac{m}{w'} = \frac{rm}{ww'}$$

Remark $M \xrightarrow{\alpha} N$ R-mod homomorphism

$\Longrightarrow$ induces R_W-mod homomorphism

$$W^{-1}M \xrightarrow{W^{-1}\alpha} W^{-1}N$$
$$\frac{m}{w} \longmapsto \frac{\alpha(m)}{w}$$

Lemma $\frac{m}{w} = 0 \in W^{-1}M \iff vm = 0$ for some $v \in W$

$\iff \operatorname{ann}_R(m) \cap W \neq \emptyset$

Proof $\frac{m}{w} = \frac{0}{1} \iff v(m \cdot 1 - 0 \cdot w) = 0$ for some $v \in W$

$\iff vm = 0$ for some $v \in W$

$\iff \operatorname{ann}(m) \cap W \neq \emptyset$

Remark $M \xrightarrow{\alpha} N$ injective $\Rightarrow$ $W^{-1}M \xrightarrow{W^{-1}\alpha} W^{-1}N$ injective

$$\frac{\alpha(m)}{w} = 0 \Rightarrow u\alpha(m) = 0 \text{ for some } u \in W$$

$$\Leftrightarrow \alpha(um) = 0 \text{ for some } u \in W$$

$$\alpha \text{ injective} \Rightarrow um = 0 \text{ for some } u \in W$$

$$\Leftrightarrow \frac{m}{w} = 0$$

Lemma $N_1, \dots, N_t \subseteq M$ R-mods, $W \subseteq R$ multiplicative set

$$W^{-1}(N_1 \cap \dots \cap N_t) = W^{-1}N_1 \cap \dots \cap W^{-1}N_t \subseteq W^{-1}M$$

Proof $\subseteq$ ✓ $\quad \supseteq$

find common denominator $\frac{n_1}{w_1} = \frac{n_2}{w_2} = \dots = \frac{n_t}{w_t}$

thm Localization is exact:

$$0 \to A \to B \to C \to 0 \quad \text{ses of R-mods}$$

$$\Rightarrow \quad 0 \to W^{-1}A \to W^{-1}B \to W^{-1}C \to 0 \quad \text{ses of R-mods}$$

Corollary $W^{-1}(M/N) \cong W^{-1}M / W^{-1}N$

Prop $W \subseteq R$ multiplicatively closed set
$I \subseteq R$ ideal $\qquad p \in \operatorname{Spec}(R)$

1) $W^{-1}I \cap R = \{ r \in R \mid wr \in I \text{ for some } w \in W \}$
2) $W \cap p = \emptyset \Rightarrow W^{-1}p = p\, W^{-1}R$ is prime
3) $\operatorname{Spec}(W^{-1}R) \longrightarrow \operatorname{Spec}(R)$ is injective
   image $= \{ p \in \operatorname{Spec}(R) \mid p \cap W = \emptyset \}$

Proof 1) $W^{-1}(R/I) \cong \frac{W^{-1}R}{W^{-1}I} \Rightarrow \ker\left(R \longrightarrow W^{-1}(R/I)\right) = R \cap W^{-1}I$

2) $W \cap p = \emptyset \Rightarrow$ no element in W kills $\bar{1} \in R/p$

$\Rightarrow W^{-1}(R/p) \neq 0$, so a domain (localization of a domain is a domain)

$\Rightarrow W^{-1}R/W^{-1}p \neq 0$ domain $\Rightarrow W^{-1}p$ prime

3) Claim

$$\begin{array}{ccc} \mathrm{Spec}(W^{-1}R) & & \{p \in \mathrm{Spec}(R) \mid p \cap W = \emptyset\} \\ q & \longmapsto & q \cap R \\ W^{-1}p & \longleftarrow\!\shortmid & p \quad \text{(ok by 2)} \end{array}$$

are inverse maps.

- $\mathfrak{d} \subseteq W^{-1}R$ ideal: $\mathfrak{d} = \left(\left\{\frac{a_i}{w_i}\right\}\right) = \left(\left\{\frac{a_i}{1}\right\}\right)$

unit multiple of $\frac{a_i}{w_i}$

$\therefore \mathfrak{d} \cap R = (\{a_i\}) \Rightarrow (\mathfrak{d} \cap R)\, W^{-1}R = \mathfrak{d}$

$W \cap p = \emptyset \Rightarrow W^{-1}p \cap R = \{x \in R \mid wx \in p \text{ for some } w \in W\}$

$= \{x \in R \mid x \in p\} = p$

p prime

Corollary $R \longrightarrow R_p$ induces the following map on Spectra:

$\{q \in \mathrm{Spec}(R) \mid q \subseteq p\} \hookrightarrow \mathrm{Spec}(R)$

<u>Determinantal Trick</u> $A \in M_{n\times n}(R)$, $v \in R^{\oplus n}$, $x \in R$

If $Av = xv$, then $\det(x I_{n\times n} - A)\, v = 0$

<u>Nakayama's lemma</u> (Nakayama - Azumaya - Krull)

<u>NAK1</u> R ring

I ideal

M fg R-module

If $IM = M$, then:

1. there is $r \in 1 + I$ st $rM = 0$
2. there is $a \in I$ st $am = m$ for all M

<u>Proof</u> $M = Rm_1 + \cdots + Rm_s$

1. $m_i = a_{i1} m_1 + \cdots + a_{is} m_s \in IM$ $\qquad a_{ij} \in I$

$A = [a_{ij}]$, $v = \begin{pmatrix} m_1 \\ \vdots \\ m_s \end{pmatrix}$

$$Av = v \Rightarrow \underbrace{\det(1 \cdot I_{s\times s} - A)}_{\in R}\, v = 0$$

$$\det(I_{s\times s} - \underset{0 \bmod I}{A}) = \det(I_{s\times s}) = 1 \pmod I$$

so $r = \det(I_{s\times s} - A) \in 1 + I$ kills M.

2. take $a = 1 - r \in I$. For all $m \in M$:

$$am = (1-r)m = m - \underbrace{rm}_{=0} = m$$

<u>NAK 2</u> $(R, \mathfrak{m})$ local ring
M fg R-mod

If $M = \mathfrak{m}M$, then $M = 0$

<u>Proof</u> $M = \mathfrak{m}M \Rightarrow rM = 0$ for some $r \in \underbrace{1+\mathfrak{m}}_{\notin \mathfrak{m} \Rightarrow \text{unit}}$

$\Rightarrow \quad 1 \cdot M = 0$

$\Leftrightarrow \quad M = 0$

<u>Note</u> In fact, $M = 0 \Leftrightarrow M = \mathfrak{m}M$.

$\Leftarrow$ prop

$\Rightarrow \quad 0 = \mathfrak{m}0$

<u>NAK 3</u> $(R, \mathfrak{m})$ local ring

$N \subseteq M$ R-mods

If $M = N + \mathfrak{m}M$, then $M = N$.

<u>Proof</u> $M = N + \mathfrak{m}M$

$$M/N = \frac{N + \mathfrak{m}M}{N} = \mathfrak{m}(M/N)$$

$\xRightarrow{\text{NAK 2}} M/N = 0 \Rightarrow M = N$

<u>Note</u> $M = N + \mathfrak{m}M \Leftrightarrow M = N$

Prop $(R, \mathfrak{m})$ local ring

M fg R-module

$m_1, \dots, m_s$ generate M $\Leftrightarrow$ $\overline{m_1}, \dots, \overline{m_s}$ generate $M/\mathfrak{m}M$

Proof $(\Rightarrow)$ obvious

$(\Leftarrow)$ $N = Rm_1 + \cdots + Rm_s \subseteq M$

$$M/N = 0 \Leftrightarrow \mathfrak{m}(M/N) = M/N$$

$$\Leftrightarrow M/N = \frac{\mathfrak{m}M + N}{N}$$

$$\Leftrightarrow M = \mathfrak{m}M + N$$

$$\Leftrightarrow M/\mathfrak{m}M = \frac{\mathfrak{m}M + N}{\mathfrak{m}M}$$

$$\Leftrightarrow \overline{N} \text{ generates } M/\mathfrak{m}M$$

Remark $R/\mathfrak{m}$ is a field, and $M/\mathfrak{m}M$ is a vector space over $R/\mathfrak{m}$