NAK $(R, \mathfrak{m})$ local ring

M fg R-mod, $N \subseteq M$ R-submodule

② $\mathfrak{m}M = M \iff M = 0$

③ $M = N + \mathfrak{m}M \iff M = N$

④ $M = Rm_1 + \cdots + Rm_n \iff \overline{m_1}, \ldots, \overline{m_n}$ generate $\underbrace{M/\mathfrak{m}M}_{R/\mathfrak{m}\text{-vector space}}$

Proof of ④: $N = Rm_1 + \cdots + Rm_n$

N generates $M \iff M = N \overset{③}{\iff} M = N + \mathfrak{m}M$

$\iff$ image of N generates $M/\mathfrak{m}M$

Def $\{m_1, \ldots, m_s\} \subseteq M$ is a <u>minimal generating</u> set for M if $\{\overline{m_1}, \ldots, \overline{m_s}\}$ are a basis for the $R/\mathfrak{m}$-vector space $M/\mathfrak{m}M$

Remark these follow from facts about vector spaces:

- All minimal generating sets for M have the same number of elements
- Every set of generators contains a minimal generating set.
- Every element in M but not in $\mathfrak{m}M$ is part of a minimal generating set.

Minimal number of generators

$$\mu(M) := \dim_{R/m}(M/mM)$$

$=$ number of generators in a minimal generating set

Graded NAK

G-NAK 1 $\quad R$ $\mathbb{N}$-graded

M $\mathbb{Z}$-graded R-mod

$M_{<a} = 0$

If $M = R_+ M$, then $M = 0$

Proof $\quad \underbrace{M}_{\text{degrees} \geq a} = \underbrace{R_+ M}_{\text{degrees} \geq a+1} \Longrightarrow M = 0$

Remark this includes all fg $\mathbb{Z}$-graded R-modules

If M is fg, there is a finite generating set of homogeneous elements (take homogeneous components of any generating set)

Set $a :=$ min degree of a generator in a given generating set

$$M \subseteq \underbrace{R}_{\text{degrees} \geq 0} M_{\geq a} \subseteq M_{\geq a} \Rightarrow M_{<a} = 0$$

G-NAK 2 $\quad$ R $\mathbb{N}$-graded

R_0 field

M $\mathbb{Z}$-graded R-mod

$M_{<a} = 0$

A set of elements generates M

$\Updownarrow$

images span M/R_+M over R_0 $\qquad$ $R/R_+ \cong R_0$ field

Notes:

→ In the graded setting, we can use NAK to show some modules are finitely generated, since it gives us a concrete way to find minimal generating sets. However, in the local setting, we can use NAK only if M is already fg.

→ If k is a field, $R = \bigoplus_{i \geq 0} R_i$, $R_0 = k$, I homogeneous ideal
$\Rightarrow$ I has a minimal generating set by homogeneous elements and this is unique up to k-linear combinations.

Def M fg $\mathbb{Z}$-graded module over $R = \bigoplus_{i \geq 0} R_i$, $R_0 = k$ field

$$\mu(M) := \dim_{R/R_+} (M/R_+M)$$

Minimal primes and support:

Recall $\mathrm{Min}(I) =$ minimal primes containing I

$$V(I) = \{ p \in \mathrm{Spec}(R) \mid p \supseteq I \}$$

Exercise $$\sqrt{I} = \bigcap_{p \in \mathrm{Min}(I)} p$$

Remark $p \in \mathrm{Spec}(R) \Rightarrow \mathrm{Min}(p) = \{p\}$

$$V(I) = V(\sqrt{I}) \Rightarrow \mathrm{Min}(I) = \mathrm{Min}(\sqrt{I})$$

Special case $\mathcal{N}(R) = \sqrt{(0)} =$ nilpotent elements is the nilradical of R.

Lemma $I = P_1 \cap \cdots \cap P_n$ where $P_i \not\subseteq P_j$ for $i \neq j$

then $\mathrm{Min}(I) = \{P_1, \ldots, P_n\}$

Proof $q \supseteq I \Rightarrow q \supseteq P_1 \cap \cdots \cap P_n$

If $q \not\supseteq P_i$ for all $\Rightarrow$ can find $f_i \in P_i$, $f_i \notin q$

$f_1 \cdots f_n \notin q$ but $f_1 \cdots f_n \in P_1 \cap \cdots \cap P_n = I$

$\Rightarrow q \supseteq P_i$ for some i. therefore, $\mathrm{Min}(I) = \{P_1, \ldots, P_n\}$

Remark If $I = p_1 \cap \dots \cap p_n$ for some primes p_i,
can always delete any unnecessary components until we get left with the set of minimal primes of I

$\mathrm{Min}\, I \subseteq \{p_1, \dots, p_n\}$.

Thm R Noetherian $\Rightarrow |\mathrm{Min}(I)| < \infty$
and
$$\sqrt{I} = I_1 \cap \dots \cap I_n$$

Proof $S = \{\text{ideals } I \subseteq R \mid \mathrm{Min}(I) \text{ infinite}\}$

Suppose $S \neq \emptyset$. R Noetherian $\Rightarrow$ S has a max element J.

If J is prime then $\mathrm{Min}(J) = \{J\}$ is finite $\Rightarrow$ J not prime

But! $\mathrm{Min}(J) = \mathrm{Min}(\sqrt{J})$
$J \subseteq \sqrt{J}$
$\Rightarrow$ J is radical.

Since J is not prime, we can find $a, b \notin J$, $ab \in J$

Claim: $J = \sqrt{J + (a)} \cap \sqrt{J + (b)}$
$\subseteq$ clear.

If $f \in \sqrt{J + (a)} \cap \sqrt{J \cap (b)}$, then $\exists\, n, m$

$$f^{n+m} \in (J + (a))(J + (b)) \subseteq J^2 + Ja + Jb + (\underbrace{ab}_{\in J}) \subseteq J \Rightarrow f \in \sqrt{J} = J$$

$\mathcal{J}$ maximal in $S \Rightarrow \mathrm{Min}(\mathcal{J}+(a)), \mathrm{Min}(\mathcal{J}+(b))$ finite

$$\mathcal{J} = \underbrace{\mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_a}_{\mathcal{J}+(a)} \cap \underbrace{\mathfrak{P}_{a+1} \cap \cdots \cap \mathfrak{P}_b}_{\mathcal{J}+(b)}$$

so $\mathrm{Min}(\mathcal{J}) \subseteq \{\mathfrak{P}_1, \dots, \mathfrak{P}_b\}$ $\lightning$

$\blacksquare$

__Def__ M R-mod
$\mathrm{Supp}(M) := \{ \mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \neq 0 \}$

__Prop__ M fg R-mod

$$\mathrm{Supp}(M) = V(\mathrm{ann}(M))$$

In particular, $\mathrm{Supp}(R/I) = V(I)$

__Proof__ $M = Rm_1 + \cdots + Rm_n$

$$\mathrm{ann}(M) = \bigcap_{i=1}^{n} \mathrm{ann}(Rm_i)$$

needs M fg

$$V(\mathrm{ann}(M)) = \bigcup_{i=1}^{n} V(\mathrm{ann}(Rm_i))$$

__Claim__ $\mathrm{Supp}(M) = \bigcup_{i=1}^{n} \mathrm{Supp}(Rm_i)$

$(\supseteq)$: $(Rm_i)_{\mathfrak{P}} \subseteq M_{\mathfrak{P}}$ so $\mathfrak{P} \in \mathrm{Supp}(Rm_i) \Rightarrow (Rm_i)_{\mathfrak{P}} \neq 0 \Rightarrow M_{\mathfrak{P}} \neq 0$
$\Rightarrow \mathfrak{P} \in \mathrm{Supp}(M)$

$(\subseteq)$ $M_{\mathfrak{p}} = R_{\mathfrak{p}} \cdot \frac{m_1}{1} + \dots + R_{\mathfrak{p}} \frac{m_n}{1}$

so $\mathfrak{p} \in \mathrm{Supp}(M) \iff \mathfrak{p} \in \mathrm{Supp}(Rm_i)$ for some i

so $\mathrm{Supp}(M) = \bigcup_{i=1}^{n} \mathrm{Supp}(Rm_i)$

So we can reduce to the case when M is cyclic.

$\frac{m}{1} = 0$ in $M_{\mathfrak{p}} \iff (R \setminus \mathfrak{p}) \cap \mathrm{ann}_R(m) \neq \emptyset$

$\iff \mathrm{ann}_R(m) \not\subseteq \mathfrak{p}$

so $\mathrm{Supp}(Rm) = V(\mathrm{ann}(m))$

Remark the hypothesis that M is fg is necessary!

Lemma R ring
M R-mod
$m \in M$

$m = 0$ in M

$\iff \frac{m}{1} = 0$ in $M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$

$\iff \frac{m}{1} = 0$ in $M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{mSpec}(R)$

Proof $m \neq 0 \Rightarrow \mathrm{ann}(m) \subseteq$ a max ideal $\Rightarrow \mathrm{Supp}(Rm) = V(\mathrm{ann}(m)) \ni$ max ideal

Lemma $\quad 0 \to L \to M \to N \to 0$ ses

$$\mathrm{Supp}(M) = \mathrm{Supp}(L) \cup \mathrm{Supp}(N)$$

Proof $\quad 0 \to L_p \to M_p \to N_p \to 0$ ses

$p \in \mathrm{Supp}(L) \cup \mathrm{Supp}(N) \Rightarrow N_p$ or $L_p \neq 0 \Rightarrow M_p \neq 0 \Rightarrow p \in \mathrm{Supp}(M)$

$p \notin \mathrm{Supp}(L) \cup \mathrm{Supp}(N) \Rightarrow N_p = 0$ and $L_p = 0 \Rightarrow M_p = 0 \Rightarrow p \notin \mathrm{Supp}(M)$

Cor $\quad L \subseteq M \Rightarrow \mathrm{Supp}(L) \subseteq \mathrm{Supp}(M)$

Cor $\quad M$ fg R-mod

$M = 0$

$\Leftrightarrow M_p = 0$ for all $p \in \mathrm{Spec}(R)$

$\Leftrightarrow M_p = 0$ for all $p \in \mathrm{mSpec}(R)$

Proof $\quad \Rightarrow$ all clear.

If $m \in M$ is nonzero, then

$$\exists \text{ max ideal} \in \mathrm{Supp}(Rm) \overset{\text{Lemma}}{\subseteq} \mathrm{Supp}(M)$$

Conclusion $\quad M \neq 0 \Rightarrow \mathrm{Supp}(M) \neq \emptyset$

Sidenote about R-module maps:

R-module homomorphism $R \longrightarrow M$	$\Longleftrightarrow$	choosing $m \in M$ $1 \longmapsto m$ $(\Rightarrow x \longmapsto xm)$
R-module homomorphism $R/I \longrightarrow M$	$\Longleftrightarrow$	choosing $m \in M$ $I \subseteq \operatorname{ann}(m)$

$$\Updownarrow$$

R-module homomorphism

$$R \longrightarrow M$$

image killed by I

<u>Def</u> R ring

M R-module

$p \in \operatorname{Spec}(R)$ is an associated prime of M if

$$p = \operatorname{ann}(m) \text{ for some } m \in M$$

$$\Updownarrow$$

$$R/p \hookrightarrow M$$

$$\operatorname{Ass}(M) := \{ p \in \operatorname{Spec}(R) \mid p \text{ associated to } M \}$$

I ideal

associated primes of I $\equiv$ associated primes of R/I