Def $\quad R$ ring
$\quad M$ R-module

$p \in \operatorname{Spec}(R)$ is a minimal prime of R if

$$I \subseteq Q \subseteq P \atop Q \in \operatorname{Spec}(R) \quad \Rightarrow \quad Q = P$$

$\operatorname{Min}(I) = \{P \in \operatorname{Spec}(R) \mid P \text{ minimal prime of } I\}$

$\operatorname{Supp}(M) := \{P \in \operatorname{Spec}(R) \mid M_P \neq 0\}$ support of M

Facts $\quad \operatorname{Supp}(M) = \emptyset \iff M = 0$

$$\operatorname{Supp}(M) = V(\operatorname{ann}(M))$$

R Noetherian
M fg R-mod $\quad \Rightarrow \quad |\operatorname{Min}(I)| < \infty$

$p \in \operatorname{Spec}(R)$ is an associated prime of M if

$p = \operatorname{ann}(m)$ for some $m \in M$

$$\Updownarrow$$

$$R/p \hookrightarrow M$$

$\operatorname{Ass}(M) := \{p \in \operatorname{Spec}(R) \mid p \text{ associated to } M\}$

I ideal
associated primes of I $\equiv$ associated primes of R/I

<u>Lemma</u> $\mathrm{Ass}(R/p) = \{p\}$ $\qquad$ p prime

<u>Proof</u> $\mathrm{ann}_R(\underbrace{x+p}_{\neq 0}) = \{s \in R \mid xs \in p\} \underset{\substack{\downarrow \\ p \text{ prime}}}{=} p$

so $x \notin p$

<u>Lemma</u> R Noetherian ring
M R-module

0) Every $\mathrm{ann}(m)$ is contained in an associated prime of M

1) $\mathrm{Ass}(M) = \emptyset \iff M = 0$

2) $\bigcup_{p \in \mathrm{Ass}(M)} p = \{\text{zerodivisors on } M\}$

$:= \{x \in R \mid xm = 0 \text{ for some } m \neq 0 \text{ in } M\}$

If R and $M \neq 0$ are graded, M has an associated prime that is homogeneous

<u>Proof</u> $M = 0 \Rightarrow \mathrm{Ass}(M) = \emptyset$ by definition. Let $M \neq 0$

If 0) holds, then $M \neq 0 \Rightarrow \exists\, m \neq 0 \Rightarrow \mathrm{ann}(m) \subseteq$ ass prime (exists!) $\Rightarrow$ 1)

$\left\{ \begin{array}{l} \text{By definition, } p \in \mathrm{Ass}(M) \Rightarrow p \subseteq Z(M). \\ x \in Z(M) \Rightarrow x \in \mathrm{ann}(m) \subseteq \text{ass prime} \end{array} \right. \Rightarrow$ 2)

Proof of 0):

$M \neq 0 \Rightarrow \quad S = \{\mathrm{ann}(m) \mid m \in M, m \neq 0\} \neq \emptyset$

Any element in S is contained in a maximal element of S (because R is Noetherian!)

Let $I = \mathrm{ann}(m)$ be a maximal element in S

$rs \in I,\ s \notin I \Rightarrow sm \neq 0 \Rightarrow \underbrace{ann(sm)}_{\in S} \supseteq \underbrace{ann(m)}_{\max \text{ in } S}$

$\Rightarrow \quad ann(m) = ann(sm)$

so $r(sm) = \underbrace{(rs)}_{\in ann(m)} m = 0 \Rightarrow r \in ann(sm) = ann(m)$

$\Rightarrow r \in I$

$\therefore$ I is prime. $\Rightarrow$ $I \in Ass(M)$

<u>Graded case</u> : $\{ ann(m) \mid m \neq 0$ is a homogeneous element $\}$

m homogeneous

$fm = 0$

$f = \underbrace{f_{a_1} + \cdots + f_{a_n}}_{\text{homogeneous}}$ $\Rightarrow$ $\underbrace{f_{a_1} m + \cdots + f_{a_n} m}_{\text{homogeneous}} = 0 \Rightarrow$ all $f_{a_i} m = 0$

$\therefore$ the annihilators of homogeneous elements are homogeneous

Repeat the argument for the annihilators of homogeneous elements

Need: <u>Lemma</u> R $\mathbb{Z}$-graded, I an ideal satisfying for any homogeneous $r, s \in R$ $\quad rs \in I \Rightarrow r \in I$ or $s \in I$

then I is prime.

Lemma $\quad 0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0 \quad$ ses

$$\mathrm{Ass}(L) \subseteq \mathrm{Ass}(M) \subseteq \mathrm{Ass}(L) \cup \mathrm{Ass}(M)$$

Proof $\quad p \in \mathrm{Ass}(L) \Longrightarrow R/p \subseteq L \underset{\text{assumption}}{\subseteq} M \Longrightarrow p \in \mathrm{Ass}(M)$

$p \in \mathrm{Ass}(M)$, say $p = \mathrm{ann}(m)$

$\Longrightarrow p \subseteq \mathrm{ann}(rm)$ for all $r \in R$

Option 1 $\quad rm \in L$ for some $r \notin p$

$a(rm) = 0 \Longleftrightarrow (ar)m = 0 \Longrightarrow ar \in p \Longrightarrow a \in p$

so $\mathrm{ann}(rm) = p$

$\therefore p \in \mathrm{Ass}(N)$

Option 2 $\quad rm \notin L$ for all $r \notin p$

Let $n :=$ image of m in N

$$\left.\begin{cases} pm = 0 \Longrightarrow pn = 0 \Longrightarrow p \subseteq \mathrm{ann}(n) \\ rn = 0 \Longrightarrow rm \in L \Longrightarrow r \in p \end{cases}\right. \Longrightarrow \mathrm{ann}(n) = p$$

$\therefore p \in \mathrm{Ass}(N)$

$\blacksquare$

Note: Both these inclusions may be strict.

theorem R Noetherian M fg R-module

there exists a (prime) filtration of M

$$M = M_t \supsetneq M_{t-1} \supsetneq M_{t-2} \supsetneq \cdots \supsetneq M_1 \supsetneq M_0 = 0$$

where $M_i/M_{i-1} \cong R/p_i$ for some primes $p_i \in \operatorname{Spec}(R)$

If R and M are $\mathbb{Z}$-graded, there exists a filtration with

$M_i/M_{i-1} \cong R/p_i\,(t_i)$ for homogeneous primes p_i, $t_i \in \mathbb{Z}$

Proof If $M \neq 0$, then $\operatorname{Ass}(M) \neq 0$, so $\exists\, R/p_1 \underset{\cong}{\hookrightarrow} M_1 \subseteq M$

If $M/M_1 \neq 0$, then $\operatorname{Ass}(M) \neq 0$, so $\exists\, R/p_2 \cong \frac{M_2}{M_1} \subseteq R/M_1$

Continue this process: $M_0 = 0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots$

M Noetherian $\Longrightarrow$ M Noetherian $\Longrightarrow$ chain stops.

Graded case: if $p_i = \operatorname{ann}(m_i)$, m_i homogeneous of degree t_i

$\left(R/p_i\right)(t_i) \xrightarrow{\cong} R m_i$ is degree preserving $\blacksquare$

Note to find m_i with $\operatorname{ann}(m_i) = p_i$, look at $0 :_{M_i} p_i$

Our m_i is in there somewhere

Corollary R Noetherian M fg R-module

$M = M_t \supsetneq M_{t-1} \supsetneq \cdots \supsetneq M_1 \supsetneq M_0 = 0$ prime filtration with $M_i/M_{i-1} \cong R/p_i$

then:

1) $\operatorname{Ass}(M) \subseteq \{p_1, \ldots, p_t\} \Rightarrow |\operatorname{Ass}(M)| < \infty$

2) M graded $\Rightarrow \operatorname{Ass}(M)$ is a finite set of homogeneous primes

Proof Just need to show $\operatorname{Ass}(M) \subseteq \{p_1, \dots, p_t\}$

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0 \quad \text{ses}$$

$$\Rightarrow \operatorname{Ass}(M_i) \subseteq \operatorname{Ass}(M_{i-1}) \cup \operatorname{Ass}(M_i/M_{i-1})$$

$$= \operatorname{Ass}(M_{i-1}) \cup \{p_i\}$$

(skip proof)

theorem Associated primes localize: R Noetherian, M R-mod

$$\operatorname{Ass}_{W^{-1}R}(W^{-1}M) = \{W^{-1}p \mid p \in \operatorname{Ass}(M),\ p \cap W = \emptyset\}$$

Proof $p \in \operatorname{Ass}(M)$, $p \cap W = \emptyset \Rightarrow W^{-1}p$ prime in $W^{-1}R$

$$0 \to R/p \to M \quad \Longrightarrow \quad 0 \to \underset{\substack{\cong \\ W^{-1}R/W^{-1}p}}{W^{-1}(R/p)} \to W^{-1}M$$

$\Rightarrow W^{-1}p \in \operatorname{Ass}(M)$

$Q \in \operatorname{Ass}(W^{-1}M) \rightsquigarrow Q = W^{-1}p$ for some $p \in \operatorname{Spec} R$, $p \cap W = \emptyset$

R Noetherian $\Rightarrow p = (f_1, \dots, f_n) \Rightarrow Q = \left(\frac{f_1}{1}, \dots, \frac{f_n}{1}\right)$

$$Q = \operatorname{ann}\left(\frac{x}{w'}\right) = \operatorname{ann}\left(\frac{x}{1}\right)$$

$$\Rightarrow \frac{f_i}{1} \cdot \frac{x}{1} = \frac{0}{1} \Rightarrow w_i f_i x = 0 \text{ for some } w_i \in W$$

$$\Rightarrow w f_i x = 0 \text{ for all } i, \quad w = w_1 \cdots w_n$$

$$\Rightarrow p(wx) = 0$$

Claim: $p = \mathrm{ann}(wx)$

Why? If $v \in \mathrm{ann}(wx)$, then

$$vwx = 0 \iff w(vx) = 0 \iff \frac{vx}{1} = \frac{0}{1} \text{ in } W^{-1}R$$

$$\Rightarrow \frac{v}{1} \in \mathrm{ann}\left(\frac{x}{1}\right) = W^{-1}p \Rightarrow tv \in p \text{ for some } t \in W$$

$$\underset{p \text{ prime}}{\overset{t \notin p}{\Longrightarrow}} v \in p \qquad !$$

$\therefore \quad p \in \mathrm{Ass}(M)$

$\blacksquare$

Corollary R Noetherian, M R-module

1) $\mathrm{Supp}(M) = \bigcup_{p \in \mathrm{Ass}(M)} V(p)$

2) M fg $\Rightarrow \mathrm{Min}(\mathrm{ann}(M)) \subseteq \mathrm{Ass}(M)$

In particular, $\mathrm{Min}(I) \subseteq \mathrm{Ass}(R/I)$

Proof 1) $p \in \mathrm{Ass}(M)$, say $p = \mathrm{ann}(m)$

$q \in V(p) \Rightarrow p_q \subseteq q_q$, and $R_q/p_q \neq 0 \Rightarrow q \in \mathrm{Supp}(R/p)$

$$\underset{\text{exact}}{0 \to R/p \to M} \quad \rightsquigarrow \quad 0 \to \underbrace{(R/p)_q}_{\neq 0} \to M_q \text{ exact}$$

$\Rightarrow q \in \mathrm{Supp}(M)$

$$q \notin \bigcup_{p \in \mathrm{Ass}(M)} V(p) \implies q \not\supseteq p \text{ for all } p \in \mathrm{Ass}(M)$$

$$\implies p \cap (R \setminus q) \neq \emptyset \text{ for all } p \in \mathrm{Ass}(M)$$

$$\implies \mathrm{Ass}_{R_q}(M_q) = \emptyset$$

$$\implies M_q = 0$$

2) $V(\mathrm{ann}(M)) = \mathrm{Supp}(M) = \bigcup_{p \in \mathrm{Ass}(M)} V(p)$

$\Rightarrow$ minimal elements agree

$\Rightarrow$ the minimal primes of $\mathrm{ann}(M)$ are all in $\bigcup_{p \in \mathrm{Ass}(M)} V(p)$

$\Rightarrow$ minimal primes must be associated

$\therefore$ P minimal prime of M (over $\mathrm{ann}(M)$) $\Rightarrow$ $P \in \mathrm{Ass}(M)$

Associated primes that are not minimal $\equiv$ embedded

Prime Avoidance

R any ring J ideal in R

$I_1, \dots, I_n$ ideal in R, I_i prime for $i > 2$

$$J \not\subseteq I_i \text{ for all } i \Rightarrow J \not\subseteq \bigcup_{i=1}^{n} I_i$$

$$\Updownarrow$$

$$J \subseteq \bigcup_{i=1}^{n} I_i \quad \Rightarrow \quad J \subseteq I_i \text{ for some } i$$

Proof Induction on n

$n = 1 \rightarrow$ nothing to show

$n \geq 2$: by induction, can find

$$a_i \notin \bigcup_{j \neq i} I_j \qquad a_i \in J \qquad \text{for each } i$$

If $a_i \notin I_i$, we are done. So assume $a_i \in I_i$ for all i

$$a := \underbrace{a_n}_{\notin I_i} + \underbrace{a_1 \cdots a_{n-1}}_{\in I_i,\ i<n} \in J \quad \Longrightarrow \quad a \notin I_i \text{ for all } i < n$$

$$\underbrace{a}_{\in I_n} = \underbrace{a_n}_{\in I_n} + \underbrace{a_1 \cdots a_{n-1}}_{\in I_n} \qquad \text{Suppose } a \in I_n \Rightarrow a_1 \cdots a_{n-1} \in I_n$$

$(\Leftarrow)$

$n = 2 \iff a_1 \in I_2$ ↯

$n > 2 \Rightarrow I_n$ prime $\Rightarrow a_i \in I_n$ for some i ↯

$\Longrightarrow a \notin I_n$

$\therefore a \notin I_i$ for all i

Graded prime avoidance

R $\mathbb{N}$-graded

I_i, J all homogeneous

$J \not\subseteq I_i$ for all $i \Rightarrow$ there exists a homogeneous element in J not in $\bigcup_{i=1}^{n} I_i$