Primary Decomposition

Fundamental theorem of Arithmetic

For every $n \in \mathbb{Z}$ there exist primes p_i and integers $n_i \geq 1$ st

$$n = \pm\, p_1^{n_1} \cdots p_k^{n_k}$$

and this product is unique up to order and signs.

Example In $\mathbb{Z}[\sqrt{-5}]$,

$$6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$$

are two <u>different</u> ways to write 6 as a product of irreducibles

An ideal I is **primary** if

$$xy \in I \Rightarrow x \in I \text{ or } y \in \sqrt{I}.$$

<u>Remark</u> $\sqrt{\text{primary}} = \text{prime}$

I is **P-primary** if $\sqrt{I} = P$

<u>Examples</u> a) Prime ideals are primary

b) In $\mathbb{Z}$, the primary ideals are (0) and (p^n), $n \geq 1$, p prime.

c) In any UFD, (f) is primary $\Leftrightarrow f = p^n$, p prime

d) $R = k[x,y,z]$, $I = (x^2, xy, y^2) = (x,y)^2$ is primary

Note $\sqrt{}$ = prime $\not\Rightarrow$ primary

Example $q = (x^2, xy) \subseteq R = k[x,y,z]$

$\sqrt{q} = (x)$ but $xy \in q$, $x \notin q$, $q \notin \sqrt{q}$

Prop R Noetherian. TFAE:

1) q is primary
2) Every zerodivisor in R/q is nilpotent
3) $\mathrm{Ass}(R/q)$ is a singleton
4) q has exactly one minimal prime, and no embedded primes.
5) $\sqrt{q} = p$ is prime and for all $r \in R$, $w \notin p$, $rw \in q \Rightarrow r \in q$
6) $\sqrt{q} = p$ is prime, and $qR_p \cap R = q$

Proof (1) $\Leftrightarrow$ (2) $y \in Z(R/q) \Leftrightarrow \exists x \notin q, xy \in q$

$\left(y \in Z(R/q) \Leftrightarrow y^n \in q\right) \Leftrightarrow \left((xy \in q, x \notin q) \Leftrightarrow (y^n \in q)\right)$

(2) $\Leftrightarrow$ (3)

$$\bigcup_{\mathrm{Ass}(R/q)} p = Z(R/q) \overset{(2)}{=} N(R/q) = \bigcap_{p \in \mathrm{Min}(R/q)} p = \bigcap_{p \in \mathrm{Ass}(R/q)} p$$

Happens $\Leftrightarrow$ there is only one prime

(3) $\Leftrightarrow$ (4) duh.

(1) $\Leftrightarrow$ (5) Rewording of the definition, since $\sqrt{\text{primary}} = \text{prime}$

(5) $\Leftrightarrow$ (6) $qR_p \cap R = \{ x \in R \mid xs \in q \text{ for some } s \notin p \}$

$$\overset{\shortparallel}{q} \Leftrightarrow \left(\begin{matrix} xs \in q \\ s \notin p \end{matrix} \Rightarrow x \in q \right)$$

<u>Remark</u> R Noetherian

$$\sqrt{I} = \text{maximal} \Rightarrow I \text{ primary}$$

$$\emptyset \neq \mathrm{Ass}(R/I) \subseteq \mathrm{Supp}(R/I) = V(I) = \{m\} \Rightarrow \mathrm{Ass}(R/I) = \{m\}$$

$\therefore$ I primary

<u>Example</u> $R = k[x,y,z]/(xy - z^n)$, k field, $n \geq 2$

$P = (x, z)$ in R

$$\left\{ \begin{matrix} xy = z^n \in P^n \\ y \notin P \\ x \notin P^n \end{matrix} \right. \Longrightarrow P^n \text{ <u>not</u> primary}$$

<u>Note</u> $R \xrightarrow{f} S$ ring map, Q primary in S

$Q \cap R$ is primary

<u>Lemma</u> $I_1, \dots, I_t$ ideals

$$\mathrm{Ass}\left(R/\bigcap_{j=1}^{t} I_j\right) \subseteq \bigcup_{j=1}^{t} \mathrm{Ass}(R/I_j)$$

In particular, the intersection of P-primary ideals is P-primary

<u>Proof</u> $0 \to R/I \cap J \longrightarrow R/I \oplus R/J$

$$\Longrightarrow \quad \mathrm{Ass}(R/I \cap J) \subseteq \mathrm{Ass}(R/I) \cup \mathrm{Ass}(R/J)$$

<u>Primary Decomposition</u> of I is

$$I = q_1 \cap \cdots \cap q_n$$

where q_i are primary.

A primary decomposition is <u>minimal</u> if

- $q_i \not\supseteq q_j$ for all $i \neq j$ ($\Leftrightarrow$ no q_i can be deleted)
- $\sqrt{q_i} \neq \sqrt{q_j}$

<u>Note</u> Any primary decomposition can be turned into a minimal one.

<u>Example</u> In $\mathbb{Z}$,

$$n = \pm p_1^{n_1} \cdots p_k^{n_k} \Rightarrow (n) = (p_1^{n_1}) \cdots (p_k^{n_k}) = \underbrace{(p_1^{n_1}) \cap \cdots \cap (p_k^{n_k})}_{\text{primary decomposition}}$$

unique minimal!

thm (Lasker, 1905, Noether, 1921)

Every ideal in a Noetherian ring has a primary decomposition

Proof Monday, March 1st, 2021, exactly 100 years after Noether's paper was published

Example (Minimal primary decompositions are not unique)

$R = k[x, y]$ $\quad$ k field

$$I = (x^2, xy) = (x) \cap (x^2, xy, y^2) = (x) \cap (x^2, y)$$

Both are minimal primary decompositions. In fact, so are

$$I = \underbrace{(x)}_{\text{shows up in all!}} \cap \underbrace{(x^2, xy, y^n)}_{\text{always has radical } (x,y)}$$