Q is primary if

$$a \notin Q,\ b \notin \sqrt{Q} \Rightarrow ab \notin Q$$

$$\Updownarrow$$

$$\mathrm{Ass}(R/Q) = \{\sqrt{Q}\}$$

$$\Updownarrow$$

Q has no embedded primes and only <u>one</u> minimal prime.

<u>Fact</u> the contraction of a primary ideal is primary

A <u>minimal</u> primary decomposition of I is

$$I = q_1 \cap \cdots \cap q_k$$

where the q_i are primary, and

- $q_i \not\supseteq q_j$ for all $i \neq j$ $\Leftrightarrow$ no q_i can be deleted
- $\sqrt{q_i} \neq \sqrt{q_j}$ for all $i \neq j$

A primary decomposition can always be turned into a minimal one.

<u>Example</u> : minimal primary decompositions are not unique

eg $I = (x) \cap (x^2, xy, y^n)$ for all $n \geq 1$

is a minimal primary decomposition

<u>thm</u> (Lasker, 1905, Noether, 1921)

Every ideal in a Noetherian ring has a primary decomposition

<u>Proof</u> I irreducible $\equiv$ not the intersection of larger ideals

<u>Claim</u> R Noeth $\Rightarrow$ every ideal is a finite intersection of irreducibles

$S = \{$ideals that are not finite intersection of irreducibles$\}$

$S \neq \emptyset \overset{R \text{ Noeth}}{\Longrightarrow} S$ has a max element I

I not irreducible $\Rightarrow I = J \cap K$, $I \subsetneq J$, $I \subsetneq K$

$\Longrightarrow J, K \notin S \Longrightarrow I =$ intersection of irreducibles $\lightning$

<u>Claim</u> I irreducible $\Longrightarrow$ primary

q not primary $\Rightarrow \exists\, xy \in q,\ x \notin q,\ y \notin \sqrt{q}$

$(q : y) \subseteq (q : y^2) \subseteq (q : y^3) \subseteq \cdots$ stops for some n.

so $y^{n+1} f \in q \Rightarrow y^n f \in q$

Will show: $(q + (y^n)) \cap (q + (x)) = q$

$\supseteq \checkmark$

$a \in (q + (y^n)) \cap (q + (x)) \Rightarrow a = \underset{\underset{q}{\cap}}{c} + b y^n$

$a \in (q + (x)) \Rightarrow ay \in q + (xy) \subseteq q$

$$\underbrace{ay}_{\in q} = (c+by^n)y = \underbrace{cy}_{\in q} + by^{n+1}$$

$$\Longrightarrow by^{n+1} \in q \Longrightarrow by^n \in q \Longrightarrow a = c + by^n \in q$$

$\therefore$ q is not irreducible. $\square$

<u>First Uniqueness theorem</u> R Noetherian

If $I = q_1 \cap \cdots \cap q_n$ is a minimal primary decomposition then $\{\sqrt{q_1}, \dots, \sqrt{q_n}\} = \mathrm{Ass}(R/I)$.

<u>Proof</u> $\mathrm{Ass}(R/I) \subseteq \bigcup_{i=1}^{n} \mathrm{Ass}(R/q_i) = \{\sqrt{q_1}, \dots, \sqrt{q_n}\}$

Need to show: $p_i = \sqrt{q_i} \in \mathrm{Ass}(R/I)$

Fix j $\quad I_j = \bigcap_{i \neq j} q_i \supsetneq I$

$I_j/I \neq 0 \Longrightarrow \exists\, a \in \mathrm{Ass}(I_j/I)$

Fix $x_j \in I_j$ $\quad a = \mathrm{ann}(x_j + I)$

$$q_j x_j \subseteq q_j \left(\bigcap_{i \neq j} q_i\right) \subseteq q_1 \cap \cdots \cap q_n = I$$

$$\Rightarrow q_j \subseteq a \underset{\text{Min}(q_j)=\{p_j\}}{\Longrightarrow} p_j \subseteq a$$

If $r \in a$, $r x_j \subseteq I \subseteq q_j \xrightarrow[q_j\ p_j\text{-primary}]{x_j \notin q_j} r \in p_j$

$\therefore$ $a = p_j$ and $\text{Ass}(I_j/I) = \{p_j\}$

$$\Longrightarrow p_j \in \text{Ass}(R/I)$$

<u>Second Uniqueness theorem</u> R Noetherian, $I \subseteq R$ ideal

In any minimal primary decomposition of I, the <u>minimal</u> components (whose radical is in $\text{Min}(I)$) are unique and equal to $q_i = I R_{p_i} \cap R$ for each $p_i \in \text{Min}(I)$

<u>Proof</u> $I = q_1 \cap \cdots \cap q_n$ any decomposition

$p_i = \sqrt{q_i}$

$$q_i R_{p_j} = \begin{cases} R_{p_j} & \text{if } q_i \not\subseteq p_j \\ \text{a primary ideal} & \text{if } q_i \subseteq p_j \end{cases}$$

$$I_{p_i} = (q_1)_{p_i} \cap \cdots \cap (q_n)_{p_n}$$

is a primary decomposition (maybe not minimal)

Suppose $p_i \in \text{Min}(I)$. then $I_{p_i} = (q_i)_{p_i}$

$$\Rightarrow I_{p_i} \cap R = q_i R_{p_i} \cap R = q_i$$

Geometric picture $\quad I = Q_1 \cap \cdots \cap Q_k$

$$\Downarrow$$

$$\begin{aligned} Z(I) &= Z(Q_1) \cup \cdots \cup Z(Q_k) \\ &= Z(\sqrt{Q_1}) \cup \cdots \cup Z(\sqrt{Q_k}) \end{aligned}$$

(whether or not I is radical!)

A primary decomposition once again recovers our decomposition of varieties into unions of irreducible varieties

Note that if P_i is an embedded prime, $P_i \supseteq P_j$ minimal prime so $Z(P_i) \subseteq Z(P_j)$ $\rightsquigarrow$ at the end of the day we only "see" the minimal components

So if P is prime, P **is** a primary decomposition for P

P^n might not be primary. But it does have a unique minimal prime, P. the n-th symbolic power of P is

$$\begin{aligned} P^{(n)} &:= \text{unique minimal component in a primary decomposition of } P^n \\ &= P^n R_P \cap R \\ &= \{ f \in R \mid s f \in P^n \text{ for some } s \notin P \} \\ &= \text{unique smallest } P\text{-primary ideal containing } P^n \end{aligned}$$

If $I = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_k$ is a radical ideal,

$$I^{(n)} := \mathfrak{P}_1^{(n)} \cap \cdots \cap \mathfrak{P}_k^{(n)}$$

$$= \bigcap_{i=1}^{k} \left(\mathfrak{P}_i^n R_{\mathfrak{P}_i} \cap R \right)$$

$$= \bigcap_{i=1}^{k} \left(I^n R_{\mathfrak{P}_i} \cap R \right)$$

= collect all the minimal components of I^n throw away all the embedded ones

<u>Note</u> $\mathfrak{P}^n = \mathfrak{P}^{(n)} \iff \mathfrak{P}^n$ is primary

<u>Geometric Interpretation</u>

Roughly speaking, if I is a radical in $R = \mathbb{C}[x_1, \dots, x_d]$

$$I^{(n)} = \{ f \in R \mid f \text{ vanishes to order } n \text{ along } Z(\ ) \}$$

(Zariski–Nagata theorem)

An Application: <u>Krull's Intersection theorem</u>

(R, m) Noetherian local ring $\Longrightarrow \bigcap_{n \geq 1} m^n = 0$

uses:

<u>lemma</u> I fg $\Rightarrow \sqrt{I}^n \subseteq I$ for some n

Proof of Krull's Height theorem

Set $J := \bigcap_{n=1}^{\infty} m^n$

$mJ = q_1 \cap \cdots \cap q_k$ primary decomposition

Claim $J \subseteq q_i$ for all i

- If $\sqrt{q_i} = m$, then $m^n \subseteq q_i$ for some $n \Rightarrow J \subseteq m^n \subseteq q_i$
- If $\sqrt{q_i} \neq m$, then $\exists x \in m$, $x \notin \sqrt{q_i}$

$$xJ \subseteq mJ \subseteq q_i \xRightarrow[q_i \text{ primary}]{x \notin \sqrt{q_i}} J \subseteq q_i$$

$$\therefore J \subseteq q_1 \cap \cdots \cap q_k = mJ \subseteq J \Rightarrow J = mJ \xRightarrow{\text{NAK}} J = 0$$

Dimension theory

A chain of primes $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ has length n

the **height** of a prime P is

$$ht(P) = \sup\{n \mid P_0 \subsetneq \cdots \subsetneq P_n = P \text{ is a chain of primes}\}$$

the (Krull) **dimension** of R is

$$\dim(R) := \sup\{n \mid \exists \text{ chain of primes of length } n \text{ in } R\}$$
$$= \sup\{ht(P) \mid P \in \operatorname{Spec}(R)\}$$
$$= \sup\{ht(m) \mid m \in \operatorname{Spec}(R)\}$$

the **height** of an ideal I is

$$\mathrm{ht}(I) := \inf\{ \mathrm{ht}(p) \mid p \in \mathrm{Min}(I) \}$$

Remarks

- $\dim(R/p) = \sup\{ n \mid q_0 \subsetneq q_1 \subsetneq \cdots \subsetneq q_n ,\ q_i \in V(p) \}$
- $\dim(R/I) = \sup\{ n \mid q_0 \subsetneq \cdots \subsetneq q_n ,\ q_i \in V(I) \}$
- $\dim(W^{-1}R) \leq \dim(R)$
- $\dim(R_P) = \mathrm{ht}(P)$
- $\dim(R) = \sup\{ \dim(R/p) \mid p \in \mathrm{Min}(R) \}$
- p prime $\Rightarrow \dim(R/p) + \mathrm{ht}(p) \leq \dim(R)$
- $\dim(R/I) + \mathrm{ht}(I) \leq \dim(R)$
- $\mathrm{ht}(0) = 0$
- $\mathrm{ht}(P) = 0 \iff P \in \mathrm{Min}(I)$