Last time

the height of a prime $\mathfrak{P}$ is

$$ht(\mathfrak{P}) = \sup\{n \mid \mathfrak{P}_0 \subsetneq \cdots \subsetneq \mathfrak{P}_n = \mathfrak{P} \text{ is a chain of primes}\}$$

the (Krull) dimension of R is

$$\dim(R) := \sup\{n \mid \exists \text{ chain of primes of length } n \text{ in } R\}$$
$$= \sup\{ht(\mathfrak{P}) \mid \mathfrak{P} \in Spec(R)\}$$
$$= \sup\{ht(\mathfrak{m}) \mid \mathfrak{m} \in Spec(R))$$

the height of an ideal I is

$$ht(I) := \inf\{ht(p) \mid p \in Min(I)\}$$

Remarks

- $\dim(R/p) = \sup\{n \mid q_0 \subsetneq q_1 \subsetneq \cdots \subsetneq q_n,\ q_i \in V(p)\}$
- $\dim(R/I) = \sup\{n \mid q_0 \subsetneq \cdots \subsetneq q_n,\ q_i \in V(I)\}$
- $\dim(W^{-1}R) \leq \dim(R)$
- $\dim(R) = \sup\{\dim(R/p) \mid p \in Min(R)\}$
- $\dim(R_{\mathfrak{P}}) = ht(\mathfrak{P})$

- p prime $\Rightarrow$ $\dim(R/p) + ht(p) \leq \dim(R)$
- $\dim(R/I) + ht(I) \leq \dim(R)$
- $ht(0) = 0$
- $ht(P) = 0 \Leftrightarrow P \in Min(R)$

Examples

1) $\dim k = 0$

2) $\dim R = 0 \Leftrightarrow$ every prime is maximal / minimal

3) $\dim \mathbb{Z} = 1$

4) R UFD $\quad ht(P) = 1 \Leftrightarrow P = (f)$, f prime

5) $\dim(k[x_1, \dots, x_d]) \geq d$:

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \dots, x_d)$$

later we will show: $\dim(k[x_1, \dots, x_d]) = \dim(k[[x_1, \dots, x_d]]) = d$

so $\dim\left(\frac{k[x_1, \dots, x_d]}{I}\right) \leq d$

the dimension of an R-module M is

$$\dim(M) := \dim(R/\operatorname{ann} M)$$

M fg $\Rightarrow \sqrt{(\operatorname{ann} M)} = \operatorname{Supp}(M) \Rightarrow \dim M = \sup\{n \mid P_0 \subsetneq \cdots \subsetneq P_n,\ M_{P_i} \neq 0\}$

R is catenary if for all primes $Q \supseteq P$, all the saturated chains

$$P = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = Q$$

(saturated $\equiv$ chain cannot be extended) have the same length

R is equidimensional if every maximal ideal has the same finite height.

What can go wrong:

Example $\dfrac{k[x,y]}{(xy, xz)}$ not equidimensional

Domain $\not\Rightarrow$ equidimensional

Notes

- R domain, $\dim(R) < \infty \Rightarrow \dim(R/f) < \dim(R)$

In general, $\dim(R/(f)) < \dim(R) \Leftrightarrow f \notin \bigcup_{\substack{p \in \operatorname{Min}(R) \\ \dim(R/p) = \dim R}} p$

Also: $\quad R$ Noetherian $\not\Rightarrow \dim(R) < \infty$

$\dim(R) < \infty \not\Rightarrow R$ Noetherian

Artinian Rings

R is Artinian if every descending chain of ideals stabilizes

$\Leftrightarrow$ every nonempty set of ideals has a minimal element

An R-module is Artinian if every descending chain of submodules stabilizes.

Exercise $\quad R$ Artinian $\Rightarrow R/I$ Artinian

M Artinian $\Leftrightarrow N$ and M/N Artinian

Length of a module $\quad M$ has finite length if it has a composition series: there is a filtration

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

st

M_{i+1}/M_i is a simple module

M is simple if $N \subseteq M \Rightarrow N = 0$ or $N = M$

($\Leftrightarrow M \cong R/m$ for some maximal ideal m in R)

$\ell(M) := \inf \{ n \mid 0 \subsetneq \cdots \subsetneq M_n$ is a strict composition series$\}$

Jordan - Hölder theorem $\ell(M) < \infty$

- $N \subsetneq M \Rightarrow \ell(M) \subsetneq \ell(N)$
- any chain in M can be refined to a composition series
- all strict composition series of M have the same length

Lemma $0 \to A \to B \to C \to 0$ ses of R-modules

$$\ell(B) = \ell(A) + \ell(C)$$

Remark If $\mathfrak{m} M = 0$ for some maximal ideal $\mathfrak{m}$,

$$\ell_R(M) = \ell_{R/\mathfrak{m}}(M/\mathfrak{m}M) = \dim_{R/\mathfrak{m}}(M/\mathfrak{m}M)$$

But finite length $\not\Rightarrow$ $R/\mathfrak{m}$-vector space for some max ideal $\mathfrak{m}$

Exercise $(R, \mathfrak{m})$ local ring

$\ell(M) < \infty \Leftrightarrow \mathfrak{m}^n M = 0$ for some n and M fg

Example $M = (R/\mathfrak{m})^n$ $\ell(M) = n$ but $\mathfrak{m} M = 0$

Lemma M R-module

$\ell(M) < \infty \Leftrightarrow M$ is Artinian and Noetherian

Proof $\ell(M) < \infty \Rightarrow$ all chains have length $\leq \ell(M)$

$\Rightarrow$ M is Artinian and Noetherian

If $M \neq 0$ is Artinian and Noetherian, $S = \{ N \subsetneq M \text{ submodule} \} \neq \emptyset$
$\Rightarrow$ S has a maximal element M_1 since M is Noetherian
then M/M_1 is simple. Inductively Construct

$M \supsetneq M_1 \supsetneq \cdots$ where each quotient is simple

M Artinian $\Rightarrow$ chain stops $\Rightarrow$ $M_i = 0$ for some i

$\therefore$ $\ell(M) < \infty$

<u>Lemma</u> R Noetherian
$\ell(M) < \infty \iff M$ fg and $\dim(R/\text{ann}(M)) = 0$

<u>Proof</u> $\ell(M) < \infty \Rightarrow M$ Noetherian $\Rightarrow M$ fg

$0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n$ Composition series

$$0 \to M_i \to M_{i+1} \to \underbrace{M_{i+1}/M_i}_{\text{Ass} = R/m_i} \to 0$$

$\text{Ass}(M_1) = \{R/m_1\}$ $\xrightarrow{\text{induction}}$ $\text{Ass}(M) \subseteq \{m_1, \ldots, m_n\}$

$\Rightarrow$ all primes containing $\text{ann}(M)$ are maximal

$\Rightarrow$ $\dim(R/\text{ann}(M)) = 0$

If M is fg and $\dim(R/\mathrm{ann}(M))=0$, then all primes containing $\mathrm{ann}(M)$ are maximal

Also, M has a prime filtration

$$M = M_t \supsetneq M_{t-1} \supsetneq \cdots \supsetneq M_0 = 0 \quad \text{with } M_i/M_{i-1} \cong R/p_i$$

$$\mathrm{ann}(M)\cdot M_i = 0 \subseteq M_{i-1} \Rightarrow \mathrm{ann}(M) \subseteq (M_{i-1} : M_i) = p_i$$

$\therefore$ all p_i are maximal and the filtration is a composition series

<u>thm</u> R ring, $I \subseteq R$ ideal

$V(I) = \{m_1, \dots, m_t\} \subseteq m\mathrm{Spec}(R)$

then I has a primary decomposition $I = q_1 \cap \cdots \cap q_k$

<u>and</u> also $I = q_1 \cdots q_k$, $R/I \cong R/q_1 \times \cdots \times R/q_k$

<u>theorem</u> Let R be a ring. TFAE

1. R is Noetherian and $\dim(R) = 0$
2. R is a finite product of local Noetherian rings of dim 0
3. $\ell_R(R) < \infty$
4. R is Artinian

Proof ① $\Rightarrow$ ② Every minimal prime is maximal

$\mathrm{Spec}(R) = m\mathrm{Spec}(R) = \mathrm{Min}(R)$ finite

Apply theorem to $I = 0$

$\Rightarrow R \cong R/q_1 \times \dots \times R/q_n$

$\mathrm{Spec}(R/q_i) = \{m_i\} \Rightarrow R/q_i$ Noetherian, $\dim(R/q_i) = 0$

② $\Rightarrow$ ③ Want to show $\ell(R) < \infty$.

Sufficient to do the case (R, m) Noetherian local, $\dim R = 0$

then $\sqrt{(0)} = m \Rightarrow m^n = 0$ for some n

So $m^n \cdot R = 0 \Rightarrow \ell(R) < \infty$

③ $\Rightarrow$ ④ $\ell(R) < \infty \Rightarrow R$ Artinian R-mod $\Rightarrow R$ Artinian ring

④ $\Rightarrow$ ① R Artinian ring

p prime $\Rightarrow R/p$ Artinian

$0 \neq a \in R/p$: $(a) \supseteq (a^2) \supseteq \cdots$ stops $\Rightarrow (a^n) = (a^{n+1})$ for some n

$\Rightarrow a^n = a^{n+1} b$ for some $b \xRightarrow[\text{domain}]{R/p} ab = 1 \Rightarrow a$ unit

$\therefore R/p$ field $\Rightarrow p$ maximal $\therefore \dim(R) = 0$

$\mathrm{Spec}(R) = m\mathrm{Spec}(R) = \mathrm{Min}(R) = \{m_i\}_i$

$m_1 \supseteq m_1 \cap m_2 \supseteq m_1 \cap m_2 \cap m_3 \supseteq \cdots$ stops

$\Rightarrow \quad m_{n+1} \supseteq m_1 \cap \cdots \cap m_n \supseteq m_1 \cdots m_n$

$\underset{m_{n+1} \text{ prime}}{\Longrightarrow} m_{n+1} \supseteq m_i \underset{m_i \max}{\Longrightarrow} m_{n+1} = m_i$

$\therefore$ there are only finitely many primes in R