<u>theorem</u> Let R be a ring. TFAE

1. R is Noetherian and $\dim(R)=0$
2. R is a finite product of local Noetherian rings of dim 0
3. $\ell_R(R)<\infty$
4. R is Artinian

<u>Proof</u> <u>Have shown</u>: $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$

$4) \Rightarrow 1)$ Have shown: If R is Artinian, then:

- $\dim(R)=0$
- $\operatorname{Spec}(R) = \operatorname{mSpec}(R)$ is finite

$\overset{\text{thm}}{\Longrightarrow}$ $R \cong R/q_1 \times \cdots \times R/q_n$ $\quad (R/q_i, m_i)$ local of dim 0

R Artinian $\Rightarrow$ R/q_i Artinian

Need to show: (R,m) Artinian, $\operatorname{Spec}(R)=\{m\} \Rightarrow R$ Noetherian

$m \supseteq m^2 \supseteq m^3 \supseteq \cdots$ stops $\Rightarrow m^n = m^{n+1}$

(Note: NAK doesn't apply → don't know if m is fg)

If $m^n \neq 0 \Rightarrow S=\{I \subseteq m \mid I m^n \neq 0\} \neq \emptyset$

R Artinian $\Rightarrow$ S has a minimal element J

$J \in S \Rightarrow \exists\, x \in J,\ x m^n \neq 0 \Rightarrow (x) \in S \Rightarrow (x) = J$

$(xm)\cdot m^n = x m^{n+1} = x m^n \neq 0 \Rightarrow xm \in S$

$$x\mathfrak{m} \subseteq (x) \qquad \Longrightarrow \qquad (x) = (x)\mathfrak{m} = J$$
$$x\mathfrak{m} \in S$$

$$\begin{cases} (x) = \mathfrak{m}(x) \\ (x) \text{ f.g.} \end{cases} \xrightarrow[\text{NAK}]{} (x) = 0 \Rightarrow J = 0 \quad \lightning$$

So $\mathfrak{m}^n = 0$, and if we take n smallest possible,

$$0 = \mathfrak{m}^n \subseteq \mathfrak{m}^{n-1} \subseteq \cdots \subseteq \mathfrak{m} \subseteq R$$

the quotients $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ are killed by $\mathfrak{m} \Rightarrow R/\mathfrak{m}$-modules, and they are Artinian R-modules, so Artinian $R/\mathfrak{m}$-mods

over a field, Artinian = Noetherian = finite length

stitch composition series together to make one for R

$\Longrightarrow$ R Artinian R-mod $\Longrightarrow$ R Artinian ring $\square$

R ring $\qquad$ R Artinian $\Longleftrightarrow \begin{cases} \dim R = 0 \\ \text{Noetherian} \end{cases} \Longleftrightarrow \ell(R) < \infty$

Warning!

M module $\qquad$ M Artinian $\begin{matrix} \not\Rightarrow M \text{ Noetherian} \\ \not\Rightarrow \ell(M) < \infty \end{matrix}$

why? A module could be Artinian and <u>infinitely generated</u>

Krull's Height theorem

theorem (Krull's Principal ideal theorem)

R Noetherian ring

Every minimal prime of (f) has height at most 1.

$$(\Rightarrow \operatorname{ht}(f) \leq 1)$$

Proof Suppose $p \in \operatorname{Spec}(R)$, $\operatorname{ht}(p) > 1$, $p \in \operatorname{Min}(f)$

$$P_0 \subsetneq \cdots \subsetneq P_n = p, \quad n \geq 2$$

- Localize at p (so p is the unique maximal ideal)
- mod out by P_0

$\Downarrow$

(R, m) Noetherian local domain

$\dim R \geq 2$

$\operatorname{Min}(f) = \{m\}$

Let $q \in \operatorname{Spec}(R)$ with $0 \subsetneq q \subsetneq m$

$\overline{R} = R/(f)$ has $\dim\left(R/(f)\right) = 0 \Rightarrow \overline{R}$ Artinian

Consider $q^{(n)} = q$-primary component in q^n

$$= q^n R_q \cap R$$

$\bar{R}$ Artinian $\Rightarrow$ $\quad \mathfrak{q}\bar{R} \supseteq \mathfrak{q}^{(2)}\bar{R} \supseteq \mathfrak{q}^{(3)}\bar{R} \supseteq \cdots$ stops

so for some n $\quad \mathfrak{q}^{(n)}\bar{R} = \mathfrak{q}^{(n+1)}\bar{R}$

In R: $\quad \mathfrak{q}^{(n)} + (f) = \underset{\supseteq \text{ always}}{\mathfrak{q}^{(n+1)} + (f)}$

so

$$\mathfrak{q}^{(n)} \subseteq \mathfrak{q}^{(n+1)} + (f)$$

Any $a \in \mathfrak{q}^{(n)}$ $\quad \underbrace{a}_{\in \mathfrak{q}^{(n)}} = \underbrace{b}_{\substack{\mathfrak{q}^{(n+1)} \\ \subseteq \mathfrak{q}^{(n)}}} + fr \quad r \in R,\ b \in \mathfrak{q}^{(n+1)}$

$\Rightarrow fr \in \mathfrak{q}^{(n)}$

$$\left.\begin{array}{l} fr \in \mathfrak{q}^{(n)} \\ f \notin \mathfrak{q} \end{array}\right\} \Rightarrow r \in \mathfrak{q}^{(n)} \Rightarrow a \in \mathfrak{q}^{(n+1)} + f\mathfrak{q}^{(n)}$$

$$\mathfrak{q}^{(n)} \subseteq \mathfrak{q}^{(n+1)} + f\mathfrak{q}^{(n)}$$

$$\Downarrow$$

$$\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)} + f\mathfrak{q}^{(n)}$$

$$\Rightarrow \quad \mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)} + \mathfrak{m}\,\mathfrak{q}^{(n)} \underset{\text{NAK}}{\Longrightarrow} \mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$$

So: $\quad \bigcap_{k \geq 1}^{\infty} \mathfrak{q}^{(k)} = \mathfrak{q}^{(n)} \neq 0$

$\qquad\qquad\qquad\qquad \mathfrak{q}^{(n)} \supseteq \mathfrak{q}^n \neq 0$ because R domain!

__But__ $\displaystyle\bigcap_{m\geq 1} q^{(m)} = \bigcap_{m\geq 1}\left(q^m R_q \cap R\right) = \underbrace{\left(\bigcap_{m\geq 1} q^m R_q\right)}_{\substack{=0 \\ \text{Krull Intersection}}} \cap R$

R domain $\Rightarrow$ $0 R_q \cap R = 0$ so

$\displaystyle\bigcap_{m\geq 1} q^{(m)} = 0$ ↯ ▫

General version: need some kind of induction.

__Lemma__ R Noetherian

$p \subsetneq q \subsetneq a \ni f$

then $\exists\, q' \ni f$ $\quad p \subsetneq q' \subsetneq a$

__Proof__ $f \in p \Rightarrow f \in q = q'$ ✓ Assume $f \notin p$.

- mod out by p
- Localize at a

$\Big\}$ (R, m) local, $f \in m$, $f \neq 0$

Need: $f \in q \subsetneq m$

$P \in \operatorname{Min}(f) \xrightarrow[\text{Ideal thm}]{\text{Principal}} \operatorname{ht} P \leq 1 \xrightarrow[f\neq 0]{R \text{ domain}} \operatorname{ht} P = 1$

$\operatorname{ht} a \geq 2 \Rightarrow \dim R \geq 2 \Rightarrow$ take any min prime over f.

Krull's Height theorem

R Noetherian

$I = (f_1, \dots, f_n)$

$P \in \mathrm{Min}(I) \Rightarrow \mathrm{ht}(P) \leq n$

$(\Rightarrow \mathrm{ht}\, I \leq n)$

Proof Induction on n.

$n=0 \Rightarrow I = 0 \Rightarrow$ minimal primes have height 0

$n=1$ is Krull's Principal Ideal theorem

$n \geq 2$ $\quad p \in \mathrm{Min}(f_1, \dots, f_n)$

$P_0 \subsetneq \cdots \subsetneq P_h = p$ saturated chain

- Case 1 $\quad f_1 \in P_1$

$\Rightarrow$ apply induction hypothesis to $\overline{R} = R/f_1$

$I\overline{R} = (\underbrace{f_2, \dots, f_n}_{n-1})\overline{R} \rightsquigarrow P_1\overline{R} \subsetneq \cdots \subsetneq P_h\overline{R}$ has length $\leq n-1$

$\Rightarrow h-1 \leq n-1 \Rightarrow h \leq n$ ✓

- Case 2 $\quad f_1 \notin P_1$ $\quad$ But $f \in P_h$, so use lemma repeatedly get new chain

$P_0 \subsetneq q_1 \subsetneq \cdots \subsetneq q_{h-1} \subsetneq P_h$ $\qquad f \in q_1$. Apply Case 1

Notes

- this bound is sharp

  $\operatorname{ht}(x_1, \dots, x_d) = d$ in $k[x_1, \dots, x_d]$

If $\operatorname{ht}(f_1, \dots, f_n) = n$, $(f_1, \dots, f_n)$ is a complete intersection

- In $R = k[x, y, z]$, $\underset{1}{(xy, xz)} = \underset{1}{(x)} \cap \underset{2}{(y, z)}$ ht:

- An associated prime can have height > # generators

  $R = \dfrac{k[x,y]}{(x^2, xy)}$ $\quad I = (0)$ $\quad$ 0 generators

  $(x,y) \in \operatorname{Ass}(R/I)$ $\quad (x,y) = \operatorname{ann} x$

- Noetherianity is necessary

theorem R Noetherian, $\dim R = d$

1) If p is a prime of height h, $\exists f_1, \dots, f_h \in p$:

   $p \in \operatorname{Min}(f_1, \dots, f_h)$

2) I any ideal in R. $\exists f_1, \dots, f_{d+1} \in I$

   $$\sqrt{I} = \sqrt{(f_1, \dots, f_{d+1})}$$

3) (R, m) local / graded k-algebra, $k = R_0$,

   $\exists f_1, \dots, f_d$ $\quad \sqrt{(f_1, \dots, f_d)} = m$

__Corollary__ $(R, \mathfrak{m})$ Noetherian local ring

$$\dim(R) = \min\{n \mid \sqrt{(f_1, \dots, f_n)} = \mathfrak{m}\} \leq \mu(\mathfrak{m})$$

In particular, R has finite dimension.

__Embedding dimension__ $(R, \mathfrak{m})$ local/graded k-alg with $R_0 = k$

$$\operatorname{embdim}(R) := \mu(\mathfrak{m})$$

__Regular Local Ring__ if $\dim(R) = \operatorname{embdim}(R)$

__Corollary__ $k[\![x_1, \dots, x_d]\!]$ is a regular local ring