Some facts about dimension/height over a Noetherian ring

$\dim(k[x_1, \dots, x_d]) = d = \dim(k[[x_1, \dots, x_d]])$

Krull Height theorem $\quad ht(f_1, \dots, f_n) \leq n$

R fg algebra over a field $\equiv R \cong \dfrac{k[x_1, \dots, x_d]}{I}$

or

$R \cong \dfrac{k[x_1, \dots, x_d]}{I}$

1. R is catenary

($p \subseteq q \Rightarrow$ every saturated $p = p_0 \subsetneq \cdots \subsetneq p_n = q$ has the same length)

If R is a domain:

2. R is equidimensional

(all maximal ideals have the same finite height)

3. $\dim(R/J) = \dim(R) - ht(J)$ for all ideals J in R

Some applications:

1) $R = k[x^3, x^2y, xy^2, y^3] \subseteq k[x,y,z]$

$\dim(R) \leq 4$ because R has 4 algebra generators

$R \cong k[a,b,c,d]/(ad-bc)$

$$\dim R = \dim\left(\frac{k[a,b,c,d]}{(ad-bc)}\right) = 4 - 1$$

$k[a,b,c,d]$ domain $\Rightarrow$ $ht(ad-bc) \geq 1$

Krull height theorem $\Rightarrow$ $ht(ad-bc) \leq 1$

Application What is $Supp(I/I^2)$? in $R = \dfrac{\mathbb{C}[x,y,z]}{(xy, yz)}$

$I = (xz)$

$Supp(I/I^2) = V(ann(I/I^2))$

$ann(I/I^2) = (y, xz)$ in R

$= (y,x) \cap (y,z)$ (radical!)

Might as well think back in $Q = \mathbb{C}[x,y,z]$ (dim 3)

$$V((x,y) \cap (y,z)) = V(\underbrace{(x,y)}_{ht\,2}) \cup V(\underbrace{(y,z)}_{ht\,2})$$

$= \{(x,y), (y,z), (x,y,z-a), (x-a,y,z) \mid a \in \mathbb{C}\}$

Over, up and down

Image Criterion $R \xrightarrow{\varphi} S$ ring homomorphism
$p \in \operatorname{Spec}(R)$

$$p \in \operatorname{image}(\varphi^*) \iff \underset{\text{for some } Q \in \operatorname{Spec}(S)}{p = Q \cap R} \iff pS \cap R = p$$

Note this is unrelated to whether or not pS is prime

Example $R = k[xy, xz, yz] \xrightarrow{\varphi} S = k[x, y, z]$

Can check with Macaulay 2: $R \cong k[a, b, c]$ polynomial ring

So $I = (xy)R$ is prime!

$IS = (xy)S = (x)S \cap (y)S$

$$IS \cap R \ni \underset{\text{in } S}{(xy) \cdot (z^2)} = \underset{\text{in } R}{(xz)(yz)} \notin (xy)R$$

so: $PS \cap R \neq I \Rightarrow I \notin \operatorname{image}(\varphi^*) \Rightarrow \varphi^*$ not surjective

or: If $Q \cap R = I$, $Q \in \operatorname{Spec}(S)$

then $Q \supseteq (xy)S \Rightarrow Q \ni x$ or $Q \ni y$

$Q \cap R \supseteq (x) \cap R = (xy, xz) \supsetneq P$

or $Q \cap R \supseteq (y) \cap R = (xy, yz) \supsetneq P$

Corollary $R \subseteq S$ direct summand $\Rightarrow \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ surjective

Integral closure of ideals $x \in R$ integral over I if
$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0 \quad \text{for } a_i \in I^i$$

$\overline{I} := \{ x \in R \mid x \text{ integral over } I \}$

$s \in S$ R-algebra integral over $I \subseteq R$ if
$$s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0 \quad \text{for } a_i \in I^i$$

Lemma $R \subseteq S$ integral
$IS \cap R \subseteq \overline{I}$

Exercise $\overline{I} \subseteq \sqrt{I}$

Lying Over $R \subseteq S$ integral
$pS \cap R = p$ for all $p \in \operatorname{Spec}(R)$
so $\operatorname{Spec}(S) \longrightarrow \operatorname{Spec}(R)$ surjective

Proof Sketch: Show $\overline{I} \subseteq \sqrt{I}$. If p prime,
$pS \cap R \subseteq \overline{p} \subseteq \sqrt{p} = p$

Example $k[xy, xz, yz] \subseteq k[x,y,z]$ not integral!
Indeed, it is not mod-finite: need to add all x^n, y^n, z^n

Theorem (Incomparability) If $R \longrightarrow S$ is integral and $P \subseteq Q$ are primes in S with $P \cap R = Q \cap R$, then $P = Q$

<u>Going-Up</u> $R \longrightarrow S$ integral

$$\begin{array}{llcccl} \text{In } S: & \mathfrak{P} & & & & \mathfrak{P} \subseteq \mathfrak{Q} \\ & \cup\!| & & \overset{\exists\, \mathfrak{q}}{\rightsquigarrow} & & \cup\!| \quad\; \cup\!| \\ \text{In } R & p & \subseteq & q & & p \subseteq q \end{array}$$

<u>Going-Down</u> $R \subseteq S$ integral
R normal domain
S domain

$$\begin{array}{llccl} \text{In } S: & & \mathfrak{Q} & & \mathfrak{P} \subseteq \mathfrak{Q} \\ & & \cup\!| & \overset{\exists\, \mathfrak{P}}{\rightsquigarrow} & \cup\!| \quad\; \cup\!| \\ \text{In } R: & p \subseteq & q & & p \subseteq q \end{array}$$

<u>Consequence</u> : $R \longrightarrow S$ integral $\Longrightarrow$ $\dim R = \dim S$

<u>Noether Normalization</u> R fg k-algebra
there exist $x_1, \dots, x_t \in R$ algebraically independent over k st

$$k[x_1, \dots, x_t] \subseteq R \quad \text{is module-finite}$$

If R is a graded k-algebra, $R_0 = k$, can choose x_i homogeneous

<u>Note</u> module-finite $\Longrightarrow$ integral $\Longrightarrow$ dim R = dim Noeth normalization

thm $k[x_1, \dots, x_d]$ Noether normalization of R

All maximal ideals of R have height d.

these are the tools we need to show:

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or

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1 R is catenary

($p \subseteq q \Rightarrow$ every saturated $p = p_0 \subsetneq \cdots \subsetneq p_n = q$ has the same length)

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Hilbert functions

k field
R $\mathbb{N}$-graded k-algebra
M $\mathbb{Z}$-graded R-module

$H_M(t) := \dim_k(R_t)$ Hilbert function of M

$h_M(z) := \sum_{i\in\mathbb{Z}} H_M(i)\, z^i$ Hilbert series of M

Example $R = k[x,y]/(x^2,y^3)$ standard graded

$$= \underset{R_0}{k} \oplus \underset{R_1}{(kx \oplus ky)} \oplus \underset{R^2}{(kxy \oplus ky^2)} \oplus \underset{R^3}{kxy^2}$$

$$H_R(t) = \begin{cases} 1, & t=0,3 \\ 2, & t=1,2 \\ 0, & \text{otherwise} \end{cases}$$ eventually 0 (polynomial of deg -1)

$h_R(z) = 1 + 2z + 2z^2 + z^3$

Polynomial Ring $R = k[x_1,\dots,x\,] \Rightarrow R_t = \bigoplus_{a_1+\dots+a_n=t} k\cdot x_1^{a_1}\cdots x_n^{a_n}$

$H_R(t) = \binom{t+n-1}{n-1}$ for $t\geq 0$ polynomial of deg $n-1$

$= $ coefficent of z^t in $(1+z+\dots+z^{a_1}+\dots)\cdots(1+z+\dots+z^{a_d}+\dots)$

$h_R(t) = (1+z+z^2+\dots)^n = \dfrac{1}{(1-z)^n}$

__thm__ R fg k-algebra
$R_0 = k$
R generated by elements of degree 1
M fg graded R-mod

$H_M(t) = P_M(t)$ for $t \gg 0$ P_M polynomial of degree $\dim(M)-1$

$$P_M(t) = \frac{e}{(\dim M-1)!} t^{\dim(M)-1} + \text{lower order terms}$$

multiplicity of M

Hilbert polynomial of M

for some positive integer.

Moreover, the Hilbert series of M is of the form

$$h_M(z) = \frac{q(z)}{(z-1)^{\dim M}} \qquad \text{for some } q(1) \neq 0.$$

to actually calculate things:

__lemma__ $0 \to L \to M \to N \to 0 \implies H_M = H_L + H_N$
ses of graded R-modules by degree 0 maps

__Proof__ $0 \to L_t \xrightarrow{f} M_t \xrightarrow{g} N_t \to 0$

__Rank-Nullity thm__ $\dim M_t = \dim g(M_t) + \dim \ker g$
$= \dim M_t + \dim L_t$

Ex $\quad S = k[x_1, \dots, x_n]$

$f \in S$ homogeneous of degree d

$R = S/(f) \longrightarrow H_S(t) = ?$

$$0 \longrightarrow S(-d) \xrightarrow{\cdot f} S \longrightarrow S/(f) \longrightarrow 0$$

$$H_S(t) = H_{S(-d)}(t) + H_R(t)$$

$$H_R(t) = H_S(t) - H_S(t-d)$$

$$= \binom{t+n-1}{n-1} - \binom{t-d+n-1}{n-1}$$

$$h_R(z) = (1-z^d)\, h_R(z)$$

thm $\quad (R, m) \quad$ local Noetherian ring or graded k-algebra with $R_0 = k$, $R = k[R_1]$

the following numbers are all the same:

- $\dim(R)$

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- $\min\{d \mid \sqrt{(f_1, \dots, f_d)} = m,\ f_i$ homogeneous$\}$ (graded case $\Downarrow$ homogeneous)

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- $1 + \deg$ (Hilbert polynomial)

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- order of the pole at 1 in the Hilbert series of R.