Last time

Noetherian Rings

A ring R is noetherian if every ascending chain

$$I_0 \subseteq I_1 \subseteq \cdots$$

of ideals in R stabilizes, meaning $I_n = I_N$ for all $n \geq N$

We started proving the following:

Proposition 1.2 R ring TFAE:

1. R is noetherian
2. Every nonempty family of ideals has a maximal element
3. Every ascending chain of fg ideals of R stabilizes
4. Given any generating set S for any ideal I, I is generated by some finite subset of S
5. Every ideal in R is finitely generated

Proof (Continued)

last time we showed ① ⇔ ②, ① ⇒ ③, ③ ⇒ ④

④ ⇒ ⑤ is obvious, since 4 requires more a priori than fg

⑤ ⇒ ① $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ ascending chain of ideals

$I = \bigcup_n I_n$ is an ideal (exercise)

(Remark: only because $I_1 \subseteq I_2 \subseteq \cdots$ etc
In general, $I \cup J$ not an ideal)

I is fg $\Rightarrow$ say $I = (f_1, \dots, f_n)$. There exists N st
$f_1, \dots, f_n \in I_N \Rightarrow I = I_N = I_n$ for all $n \geq N$

Examples 1) $R = k$ a field. the only ideals in R are (0) and R, so R is noetherian

2) $\mathbb{Z}$ is a noetherian ring, since all ideals are of the form (n). In fact, any PID is noetherian

3) $\mathbb{C}\{z\} = \{ f(z) \in \mathbb{C}[[z]] : f \text{ is analytic around } 0 \}$

Every ideal is of the form $(z^n) \Rightarrow$ this is a PID

4) $R = k[x_1, x_2, x_3, \dots]$ is <u>not</u> noetherian

$$(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \cdots$$

5) $R = k[x, x^{1/2}, x^{1/3}, x^{1/4}, \dots] \subseteq k[x]$ is <u>not</u> noetherian

$$(x) \subseteq (x^{1/2}) \subseteq (x^{1/3}) \subseteq \cdots \quad \text{is infinite}$$

6) $R = C(\mathbb{R}, \mathbb{R})$ Continuous real valued functions <u>not</u> noeth

$$I_n = \{ f(x) : f|_{[-\frac{1}{n}, \frac{1}{n}]} \equiv 0 \}$$

forms an increasing chain

Same construction shows $C^\infty(\mathbb{R}, \mathbb{R})$ <u>not</u> noetherian

<u>Remark</u> Let R be a ring and $I \subseteq R$ be an ideal in R.

$$\{\text{ideals in } R/I\} \xleftrightarrow{1:1} \{\text{ideals in } R \text{ containing } I\}$$

So R noetherian $\Rightarrow$ R/I noetherian

Definition An R-module M is <u>noetherian</u> if every ascending chain

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

of submodules of M stabilizes

Remark R noetherian ring $\Leftrightarrow$ R is a noetherian R-module

However,

$R \subseteq S$ noetherian rings $\not\Rightarrow$ S noetherian R-module

Example $\mathbb{Z} \subseteq \mathbb{Q}$ are noetherian rings, but $\mathbb{Q}$ is not a noeth $\mathbb{Z}$-mod

$$0 \subseteq \mathbb{Z}\tfrac{1}{2} \subseteq \mathbb{Z}\tfrac{1}{2} + \mathbb{Z}\tfrac{1}{3} \subseteq \mathbb{Z}\tfrac{1}{2} + \mathbb{Z}\tfrac{1}{3} + \mathbb{Z}\tfrac{1}{5} \subseteq \cdots$$

does not stabilize.

Remark Nonnoetherian rings can have <u>lots</u> of noetherian R-modules

Prop M R-mod. TFAE:

1) M is a noetherian R-module
2) Every nonempty family of submodules of M has a max
3) Every ascending chain of fg submodules stabilizes
4) Given any generating set S for a submodule N, N is generated by some finite subset of S
5) Every submodule of M is fg

In particular, a noetherian module must be fg

Proof: Exercise

Exact sequence: the sequence of R-modules and R-module maps

$$\cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \xrightarrow{f_{n+1}} \cdots$$

is exact if $\operatorname{im} f_{n-1} = \ker f_n$ for all n

An exact sequence of the form

$$0 \to A \to B \to C \to 0$$

is a short exact sequence (ses)

Remarks

- $0 \to A \xrightarrow{f} B$ is exact $\Leftrightarrow$ f is injective
- $A \xrightarrow{f} B \to 0$ is exact $\Leftrightarrow$ f is surjective

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is exact $\Leftrightarrow$ $\begin{cases} f \text{ is injective} \\ \operatorname{im} f = \ker g \\ g \text{ is surjective} \end{cases}$

so:

$$A \equiv f(A) \subseteq B$$

$$C = \operatorname{im} g \cong B/\ker f = B/A$$

Lemma $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ ses of R-modules

B is noetherian $\Leftrightarrow$ A and C are noetherian

Proof $(\Rightarrow)$ $A \subseteq B$ submodule

Submodules of A are submodules of B $\Rightarrow$ A noetherian

(submodules of noetherian modules are noetherian)

Submodules of $C = B/A$ are of the form D/A for some submodule D of B (containing A), and D fg $\Rightarrow$ D/A fg.

$\Rightarrow$ A is noetherian

$(\Leftarrow)$ $0 \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ chain of submodules of B

1) $M_1 \cap A \subseteq M_2 \cap A \subseteq \cdots$ chain of submodules of A $\Rightarrow$ stabilizes

2) $g(M_1) \subseteq g(M_2) \subseteq \cdots$ chain of submodules of C $\Rightarrow$ stabilizes

Suppose they both stabilize after step n. Fix $x \in M_{n+1} \supseteq M_n$.

$g(x) \in g(M_{n+1}) = g(M_n)$.

then $g(x) = g(y)$ for some $y \in M_n (\subseteq M_{n+1})$

$$g(x-y) = 0 \iff x-y \in \ker g = A$$

So $x-y \in A$, and $x-y \in M_{n+1}$

$\Rightarrow x-y \in A \cap M_{n+1} = A \cap M_n$

$\Rightarrow \underbrace{x-y}_{\in M_n} \in M_n \Rightarrow x \in M_n$ ∎

<u>Corollary</u> A, B noetherian $\Rightarrow A \oplus B$ noetherian

<u>Proof</u> $0 \to A \to A \oplus B \to B \to 0$ ses ∎

<u>Corollary</u> M noetherian $\iff M^n$ noetherian

In particular, R Noetherian ring $\Rightarrow R^n$ noetherian R-mod

<u>Proof</u> $n=1$ duh.

Induction: $0 \to M^n \to \underbrace{M^{n+1}}_{M^n \oplus M} \to M \to 0$

<u>Prop</u> R Noetherian ring, M R-module

M Noetherian $\iff$ M fg

So M fg R-mod, $M \supseteq N \Rightarrow N$ fg R-mod

Proof ($\Rightarrow$) follows from equivalent definitions

($\Leftarrow$) M fg $\Rightarrow$ $M \cong R^n / \ker(R^n \xrightarrow{\pi} M)$

Hilbert's Basis theorem

R noetherian ring $\Rightarrow$ $R[x_1, \dots, x_n]$ is a noetherian ring

(Also $R[[x_1, \dots, x_n]]$ is noetherian)

Remark this means that

every system of polynomial equations in finitely many variables can be written in terms of finitely many equations