Last time

R is a noetherian ring $\Leftrightarrow$ every ideal in R is fg

M is a noetherian module $\Leftrightarrow$ every submodule of M is fg

thm R Noetherian ring, M R-module
M is a Noetherian R-module $\Leftrightarrow$ M is fg

Remark ($\Rightarrow$) always true, ($\Leftarrow$) false if R not noetherian

Hilbert's Basis Theorem

R Noetherian ring $\Rightarrow$ $R[x]$ is a Noetherian ring
(and so is $R[[x]]$)

Corollary $k[x_1, \dots, x_d]$ is a Noetherian ring for any field k

Rule of thumb: For nonnoetherian examples, see $R = k[x_1, x_2, \dots]$ and its quotients

Proof of Hilbert Basis:

Let $I \subseteq R[x]$ be an ideal. We will show I is fg

$J := \{ a \in R : ax^n + \text{lower order terms} \in I \text{ for some } n \} \subseteq R$

J is an ideal in R (exercise) $\Rightarrow$ J is fg, $J = (a_1, \dots, a_t)$

Suppose a_i = leading coefficient of $f_i \in I$

Set $N := \max \{\deg f_i\}$

Given any $f \in I$ of degree $> N$,

$lc(f)$ = leading term of f = combination of a_i

f - some combination of the f_i has $<$ degree f

$$\left(\begin{aligned} &lc(f) = x_1 a_1 + \cdots + x_n a_n \\ \Rightarrow\ &\deg\left(f - \sum_i x_i a_i f_i x^{\deg f - \deg f_i}\right) < \deg f \end{aligned} \right)$$

$\Rightarrow$ f - some combination of the f_i has degree $\leq N$

f = an element in I of degree $\leq N$ + an element in $(f_1, \ldots, f_t)$

$$\Rightarrow f \in I \cap (R + Rx + Rx^2 + \cdots + Rx^N) + (f_1, \ldots, f_t)$$

$R + Rx + \cdots + Rx^N$ is a fg submodule of $R[x]$

$\Rightarrow$ $I \cap (R + Rx + \cdots + Rx^N)$ is fg, say $= (f_{t+1}, \ldots, f_s)$

<u>then</u> $I = (f_1, \ldots, f_t, f_{t+1}, \ldots, f_s)$

So I is fg, and $R[x]$ is a Noetherian ring

Power series case: take J = lowest degree coeffs of elements in I

$R \subseteq S$ subring $\Rightarrow$ S is an algebra over R, meaning

S is a ring with an R-module structure satisfying

$$r(s_1 s_2) = (r s_1) s_2 \quad \text{for all } r \in R, s_1, s_2 \in S$$

More generally, given a ring homomorphism $\varphi: R \to S$,

S is an algebra over R via φ, by setting $r \cdot s := \varphi(r) s$.

$\Lambda \subseteq S$ generates S as an R-algebra if

- the only subring of S containing $\varphi(R)$ and Λ is S

$\Updownarrow$

- Every element in S is a polynomial in Λ with coefficients in $\varphi(R)$

$\Updownarrow$

- $R[X]$ polynomial ring in $|\Lambda|$ indeterminates

the ring homomorphism $R[X] \xrightarrow{x} S$ is surjective
$$x_i \longmapsto \lambda_i$$

$\varphi: R \to S$ is algebra-finite / S is a fg R-algebra

/ S of finite type over R if

S can be generated by finitely many elements as an R-alg

$$S \text{ fg } R \text{ alg} \Leftrightarrow S = R[f_1, \dots, f_t]$$

So to recap:

$$M \text{ fg } R\text{-module} \iff M \underset{\text{as } R\text{-mod}}{\cong} R^n/N \quad \text{for some } n,\ N \subseteq R^n$$

$$S \text{ fg } R\text{-algebra} \iff S \underset{\text{as a ring}}{\cong} R[x_1, \dots, x_n]/I \quad \text{for some } n,\ I \text{ ideal}$$

here I = ideal of relations of S

Remark $\varphi : R \to S$ ring homomorphism

1) φ surjective $\Rightarrow$ 1 generates S over R $\Rightarrow$ S is alg-finite over R

2) to determine if S is a algebra-finite over R, can factor φ as

$$R \twoheadrightarrow R/\ker\varphi \xrightarrow{\varphi} S \qquad \text{always}$$

so it's sufficient to consider the case when φ is injective so we identify $R \equiv \varphi(R)$

Ex Every ring is a $\mathbb{Z}$-algebra, but usually not a fg one

eg, $\mathbb{Q}$ is not a fg $\mathbb{Z}$-algebra.

Def $s_1, \dots, s_n$ are algebraically independent if there are no relations between them, so

$R[s_1, \dots, s_n] \cong$ polynomial ring in n variables over R

<u>Rem</u> $A \subseteq B \subseteq C$

1) $\begin{cases} A \subseteq B \text{ alg-fn} \\ B \subseteq C \text{ alg-fn} \end{cases} \implies A \subseteq C$ alg-fn

2) $A \subseteq C$ algebra finite $\implies B \subseteq C$ algebra-finite

<u>Ex</u> k field, $B = \underbrace{k[x, xy, xy^2, xy^3, \dots]}_{\underline{\underline{\text{not}}} \text{ alg finite over } k} \subseteq C = \underbrace{k[x,y]}_{\text{alg-fn}/k}$

Any finitely generated k-subalgebra of B is contained in $k[x, xy, \dots, xy^m]$ all the elements in $k[x, xy, \dots, xy^m]$ are linear combinations of terms of the form $x^a y^{\leq am} \implies xy^{m+1} \notin k[x, \dots, xy^m]$

<u>Corollary of Hilbert Basis thm</u>

R Noetherian $\implies$ any fg R-alg is Noetherian

In particular, if k is a field then $R = \frac{k[x_1, \dots, x_d]}{I}$ is Noetherian

<u>Remark</u> the converse is false: many Noetherian rings are <u>not</u> fg algebras over a field.

If S is an R-algebra, we can also consider its module structure over R. We say S is module-finite if it is a fg R-mod

Remark

1) $\varphi : R \to S$ surjective $\Rightarrow$ $S \cong R/\ker\varphi$ is gen by $1 \Rightarrow S$ is mod-fin

2) Suffices to study the case when φ is injective

Examples

1) $k \subseteq L$ field extension
L module-finite over k $\Rightarrow$ L is a finite field extension of k
(L is a finite dimensional k-vector space)

2) $\mathbb{Z} \subseteq \mathbb{Z}[i]$ is module-finite: every element in $\mathbb{Z}[i]$ looks like
$a+ib$, $a, b \in \mathbb{Z}$
So in fact $\mathbb{Z}[i]$ is a 2-generated $\mathbb{Z}$-module $(\{1, i\})$

3) $R \subseteq R[x]$ is algebra-finite but not module-finite
$R[x]$ is a free R-module with basis $\{1, x, x^2, x^3 \dots\}$
so $1, x, x^2, \dots$ are linearly independent but algebraically dependent.

4) $k[x] \subseteq k[x, \frac{1}{x}]$ is <u>not</u> module-finite

$$\underbrace{\frac{1}{x} \cdot k[x]}_{\substack{\text{everything of the} \\ \text{form } \frac{f(x)}{x^{\leq 1}}}} \subseteq \underbrace{\frac{1}{x^2} \cdot k[x]}_{\substack{\text{everything of} \\ \text{form } \frac{f(x)}{x^{\leq 2}}}} \subseteq \underbrace{\frac{1}{x^3} \cdot k[x]}_{\substack{\text{everything} \\ \frac{f(x)}{x^{\leq 3}}}} \subseteq \cdots$$

infinite chain!

<u>Unrelated note</u> $k[x, \frac{1}{x}]$ is <u>not</u> a field. Eg, $1-x$ does not have an inverse.

<u>Def</u> R A-alg

$r \in R$ is <u>integral</u> over A if

$$r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

for some $n \geq 1$ and $a_i \in A$