Last time S an R-algebra (may assume $R \subseteq S$)

- S is algebra-finite if $S = R[f_1, \dots, f_n]$ for some $f_i \in S$
- S is module-finite if $S = Rf_1 + \dots + Rf_n$ for some $f_i \in S$

Remark Module finite $\Rightarrow$ algebra-finite (same generating set!)

But algebra finite $\not\Rightarrow$ module finite (eg $R \subseteq R[x]$)

Remark $A \subseteq B \subseteq C$ rings

- $\left.\begin{array}{l} A \subseteq B \text{ mod-fn} \\ B \subseteq C \text{ mod-fn} \end{array}\right. \Rightarrow A \subseteq C$ mod-fn
- $A \subseteq C$ mod-fn $\Rightarrow$ $B \subseteq C$ mod-fn
- $A \subseteq C$ mod-fn $\not\Rightarrow$ $A \subseteq B$ mod-fn
  (not as easy)

Lemma $R \subseteq S$ module-finite
N fg S-module.
then N is a fg R-module by restriction of scalars

thus the composition of module-finite maps is mod-finite

Remark N is an R-module by restriction of scalars

$$\underset{R}{\underset{\cap}{r}} \cdot \underset{N}{\underset{\cap}{n}} = \underset{S}{\underset{\cap}{r}} \bullet n$$

(We are restricting our scalars in S to R)

Proof If $S = Ra_1 + \cdots + Ra_r$ and $N = Sb_1 + \cdots + Sb_s$,

$$N = \sum_{i=1}^{r} \sum_{j=1}^{s} R a_i b_j \quad \text{as an } R\text{-module.}$$

Def R A-alg

$r \in R$ is <u>integral</u> over A if

$$r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

for some $n \geq 1$ and $a_i \in A$

Rem $r \in R$ integral over $A \iff r$ integral over $\varphi(A)$

so we might as well assume φ is injective

the <u>integral closure</u> of A in R is

$$\{ r \in R \mid r \text{ integral over } A \} \subseteq R$$

Remark Integral $\underset{\not\Leftarrow}{\Rightarrow}$ algebraic

A is integrally closed in R if $x \in R$ integral over $A \Rightarrow x \in A$

Special Case If R is a domain,

$R \subseteq frac(R) =$ field of fractions of $R = \{\frac{r}{s} : s \neq 0\}$

the integral closure of R in frac(R) is denoted $\overline{R}$

R is a <u>normal domain</u> if $R = \overline{R}$

<u>Prop</u> $A \subseteq R$ rings

1) $x \in R$ integral over $A \Rightarrow A[x]$ module-finite over A

2) $x_1, \dots, x_t \in R$ integral over $A \Rightarrow A \subseteq A[x_1, \dots, x_t]$ mod-fin.

<u>Proof</u> 1) $x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$

$\Rightarrow R[x] = R + Rx + Rx^2 + \cdots + Rx^{n-1}$

Since: $x^n = -a_{n-1} x^{n-1} - \cdots - a_1 x - a_0 \in R + \cdots + Rx^{n-1}$

$x^{n+1} = x\left(-a_{n-1} x^{n-1} - \cdots - a_1 x - a_0\right)$

$\in R + \cdots + Rx^n \subseteq R + \cdots + Rx^{n-1}$

$\vdots$

inductively, we show $x^m \in R + \cdots + Rx^{n-1}$ for all m

and thus any polynomial in x is in $R + \cdots + Rx^{n-1}$

2) $\underbrace{A_0 := A \subseteq A_1}_{\text{mod-fin}} := A[x_1] \subseteq A_2 := A[x_1, x_2] \subseteq \cdots \subseteq A_t = A[x_1, \dots, x_t]$

A_{i+1} is module-finite over A_i because x_{i+1} is integral over A_i (via the same monic equation used over A)

Composition of module-finite maps is mod-finite $\Rightarrow A \subseteq A_t$ mod-fin

<u>Determinantal Trick</u>

R ring

$B \in M_{n \times n}(R)$

$v \in R^n$ (vector)

$x \in R$

1) $\operatorname{adj}(B) \cdot B = \det B \cdot I_{n \times n}$

where $\operatorname{adj}(B)_{ij} = (-1)^{i+j} \det(\hat{B}_{ij})$

matrix

delete i-th row, j-th column

2) $Bv = xv \Rightarrow \det(x I_{n \times n} - B) v = 0$

<u>Proof</u> 1) See notes

2) $Bv = xv \Rightarrow (x I_{n \times n} - B) v = 0$

$$\det(x I_{n \times n} - B) v = \operatorname{adj}(x I_{n \times n} - B) \cdot (x I_{n \times n} - B) v = 0$$

thm $A \subseteq R$ module-finite ring extension $\Rightarrow$ $A \subseteq R$ integral

Proof Let $x \in R$. We will show x is integral over A.

$$R = A r_1 + \cdots + A r_t \qquad \text{WLOG: } r_1 = 1$$

$$x r_i = \sum_{j=1}^{t} a_{ij} r_j$$

$C = [a_{ij}]$

$v = \begin{pmatrix} r_1 \\ \vdots \\ r_t \end{pmatrix}$ $\longrightarrow$ $x v = C v \xrightarrow[\text{trick}]{\det} \det(x I_{n\times n} - C) v = 0$

$$v = \begin{pmatrix} 1 \\ \vdots \\ r_t \end{pmatrix} \Rightarrow \det(x I_{n\times n} - C) = 0$$

expand as a polynomial in x $\Rightarrow$ monic equation in $x = 0$

Corollary R mod-finite over A $\Leftrightarrow$ R is alg-finite and integral over A

Proof ($\Rightarrow$) OK by previous theorem

($\Leftarrow$) $R = A[f_1, \ldots, f_t]$ integral over A

$\Rightarrow$ each f_i integral over A $\Longrightarrow$ module-finite

Corollary

- R generated as an A-algebra by integral elements $\Rightarrow$ R integral
- $A \subseteq S \Rightarrow \{s \in S : s \text{ integral over } A\}$ is a subring of S

Proof • $R = A[\Lambda]$, every $\lambda \in \Lambda$ integral over A

$r \in R \Rightarrow r$ can be written using finitely many $\lambda \in \Lambda$

$\Rightarrow r \in A[\lambda_1, \dots, \lambda_t]$ module-finite $\Rightarrow r$ integral

- $\{\text{integral elements}\} \subseteq A[\{\text{integral elements}\}] \underset{\text{by 1)}}{\subseteq} \{\text{integral}\}$

$\Rightarrow$ the set of integral elements is a ring

$\Rightarrow$ Integral Closure of A in R is a ring

Ex:

1) $\quad R = \mathbb{C}[x,y] \subseteq S = \dfrac{\mathbb{C}[x,y,z]}{(x^2+y^2+z^2)}$

S is module-finite over R, because:

- $S = R[z]$, and
- z satisfies $r^2 + x^2 + y^2 = 0 \Rightarrow z$ integral

2) Not all integral extensions are module-finite

$k = \bar{k}$ (alg closed field),

$$k[x] \subseteq R = k[\underbrace{x, x^{1/2}, x^{1/3}, \dots}_{\text{gen by integral elements over } k[x]}] \quad \underline{\underline{\text{not}}} \text{ alg-finite}$$

eg $x^{1/n}$ satisfies $r^n - x$

<u>Artin-Tate lemma</u> $\quad A \subseteq B \subseteq C$ such that:

- A is Noetherian
- C is module-finite over B
- C is algebra-finite over A

then B is <u>algebra-finite</u> over A.

Proof $C = A[f_1, \dots, f_r]$ and $C = Bg_1 + \dots + Bg_s$

$f_i = b_{i1}g_1 + \dots + b_{is}g_s$, $g_ig_j = b_{ij1}g_1 + \dots + b_{ijs}g_s$

$b_{ij}, b_{ijk} \in B$

$B_0 := A[\{b_{ij}, b_{ijk}\}] \subseteq B$

Noetherian $\Rightarrow$ B_0 Noetherian

Any $c \in C$ is a polynomial in the f_i

$\Rightarrow c \in A[\{b_{ij}\}][g_1, \dots, g_s]$

rewrite g_ig_j as linear combinations of g_i's repeatedly

$c \in B_0 g_1 + \dots + B_0 g_s$

$\therefore$ C is a fg B_0-module, $B \subseteq C$

$\Rightarrow B$ is a fg B_0-module

$\Rightarrow B_0 \subseteq B$ is mod-fn $\Rightarrow$ alg-fn

$$\underbrace{A \subseteq B_0}_{\text{alg-fn}} \underbrace{\subseteq B}_{\text{alg-fn}} \Rightarrow A \subseteq B \text{ alg-fn}$$

An application to invariant rings

Q: Given a finite set of symmetries, consider all polynomials fixed by those symmetries. Can we express all fixed polynomials in terms of finitely many?

More precisely G group acting on $R = k[x_1, \ldots, x_d]$, fixes k

$$R^G := \{ f \in R : \ g \cdot f = f \ \text{ for all } g \in G \}$$

Exercise R^G is a subring of R

Q: Is R^G a fg k-algebra?

Prop G finite $\Rightarrow$ $R^G \subseteq R$ is module-finite

Proof $k \subseteq R$ alg-fn $\Rightarrow$ $R^G \subseteq R$ alg-fn

$r \in R \rightsquigarrow F_r(t) = \prod_g (t - g \cdot r) \in R[t]$

say G fixes t $\Rightarrow$ $F_r(t) \in (R[t])^G = R^G[t]$

leading term $t^{|G|}$ $\Rightarrow$ $F_r(t)$ monic $\Rightarrow$ r integral $/ R^G$

$R^G \subseteq R$ integral and alg-fn $\Rightarrow$ mod-fn

<u>thm</u> (Noether) R^G is a finitely-generated k-algebra

<u>Proof</u> $\underbrace{k \subseteq \underbrace{R^G \subseteq R}_{\text{mod-fin}}}_{\text{alg-fin}} \xRightarrow[\text{Tate}]{\text{Artin}} k \subseteq R^G$ alg-fin.

Noeth

$\blacksquare$