So far:

R Noetherian ring $\Leftrightarrow$ Every (prime) ideal I is fg

eg, $R = \dfrac{k[x_1, \ldots, x_d]}{I}$, k field

Over a Noetherian ring R
M Noetherian R-mod $\Leftrightarrow$ M fg R-mod

$A \subseteq R$ ring extension, $f_i \in R$

$A \subseteq R$ algebra-finite $\Leftrightarrow$ $R = A[f_1, \ldots, f_n]$ $\Leftrightarrow$ $R \cong \dfrac{A[x_1, \ldots, x_n]}{I}$

$A \subseteq R$ module-finite $\Leftrightarrow$ $R = Af_1 + \cdots + Af_n$ $\Leftrightarrow$ $R \cong A^n/N$

Graded rings R is $\mathbb{N}$-graded if

$$R = \bigoplus_{n \geqslant 0} R_n$$

where $R_a R_b \subseteq R_{a+b}$ for all $a, b \geqslant 0$

More generally, if T is a monoid (has an associative operation with identity)
R is T-graded if

$$R = \bigoplus_{t \in T} R_t$$

and $R_a R_b \subseteq R_{a+b}$ for all $a, b \in T$

Most common examples use $T = \mathbb{N}$ or $\mathbb{Z}$.

$f \in R$ is homogeneous (of degree a) if $f \in R_a$, and

$$\deg f := a \qquad \text{or} \qquad |f| = a.$$

Remark Each element is a unique sum of homogeneous elements called its graded components or homogeneous components

Remark R_0 is a subring of R and $(1 \in R_0)$

Ex: a) Every ring R is trivially a graded ring via $R_0 = R$, $R_n = 0$ for $n \neq 0$

b) k field, $R = k[x_1, \dots, x_n]$

standard grading $R_d := \sum_{a_1 + \dots + a_n = d} k \cdot x_1^{a_1} \cdots x_n^{a_n}$

(k-vector space spanned by the monomials with total degree d)

$x_1^2 + x_2 x_3$ is homogeneous, $x_1^2 + x_2$ is not

c) Any choice of $(\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ gives $R = k[x_1, \dots, x_n]$ an $\mathbb{N}$-grading:

set x_i to be homogeneous of degree β_i

this is the grading with weights $(\beta_1, \dots, \beta_n)$

d) Fine grading on $R = k[x_1, \dots, x_n]$: $\mathbb{N}^n$-grading given by

$$R_{(d_1, \dots, d_n)} = k \cdot x_1^{d_1} \cdots x_n^{d_n}$$

Remark $f \in k[x_1, \dots, x_n]$

$$f(\lambda x_1, \dots, \lambda x_n) = \lambda^n f(x_1, \dots, x_n) \iff f \text{ homogeneous wrt standard grading}$$

$$f(\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n) = \lambda^{w_1 + \dots + w_n} f(x_1, \dots, x_n) \iff f \text{ homogeneous wrt some weighted grading}$$

f is quasihomogeneous

An ideal I is homogeneous if it is generated by homogeneous elements.

Equivalently, I is homogeneous if

$f \in I \iff$ every component of f is in I

So I is homogeneous $\iff$ $I = \bigoplus_n I_n$, where $I_n = I \cap R_n$

Ex $R = \bigoplus_{n \geq 0} R_n$, $R_0 = k$ field

$R_+ = \bigoplus_{n \geq 1} R_n$ is the homogeneous maximal ideal

Indeed, this is the only ideal that is both homogeneous and maximal

Lemma I homogeneous ideal in the T-graded ring R, then R/I is naturally a T-graded ring

Proof $$R/I = \frac{\bigoplus R_a}{\bigoplus I_a} = \bigoplus \frac{R_a}{I_a}$$

Ex $R = \dfrac{k[x,y,z]}{(x^2+y^3+z^5)}$ does not admit an $\mathbb{N}$-grading with

$$\deg x = \deg y = \deg z = 1$$

but it does with $\deg x = 15$, $\deg y = 10$, $\deg z = 6$.

R T-graded ring
An R-module M is a **T-graded R-module** if

$$M = \bigoplus_{a \in T} M_a$$

$$\text{and } R_a M_b \subseteq M_{a+b}$$

R, S T-graded rings
A ring homomorphism $R \xrightarrow{f} S$ is degree-preserving or graded if

$$f(R_a) \subseteq S_a \qquad \text{for all } a \in T$$

M, N graded R-modules
An R-module homomorphism $M \xrightarrow{f} N$ is **graded** of **degree** d if

$$f(M_a) \subseteq N_{a+d} \qquad \text{for all } a \in T$$

Remark Ring homomorphism $\Rightarrow$ $1_R \mapsto 1_R$ $\Rightarrow$ degree 0 (if graded)

Examples

a) Ring homomorphism:

$$k[x,y,z] \xrightarrow{f} k[s^2, st, t^2] \subseteq \underbrace{k[s,t]}_{\text{fine grading}}$$

f degree preserving $\iff$ $\deg(x) = (2,0)$, $\deg(y) = (1,1)$, $\deg(z) = (0,2)$

b) Module homomorphism:

k field, $R = k[x_1, \dots, x_n]$ standard grading

$c \in R_0$ $\quad$ $R \xrightarrow{f} R$, $r \longmapsto cr$ $\quad$ degree-preserving

$g \in R_d$ $\quad$ $R \xrightarrow{f} R$, $x \longmapsto gx$ $\quad$ degree d map

Can turn this into a degree 0 map:

$R(-d) :=$ R with grading $R(-d)_t = R_{t-d}$

$R(-d) \xrightarrow{f} R$, $r \longmapsto gr$ $\quad$ has degree 0

Earlier: R Noetherian ring

$R \subseteq S$ alg-fin $\Longrightarrow$ S Noetherian

But R Noetherian

$R \subseteq S$ Noetherian $\not\Longrightarrow$ $R \subseteq S$ alg-fin

Prop R $\mathbb{N}$-graded

$f_1, \dots, f_n \in R$ of degree > 0

$$(f_1, \dots, f_n) = R_+ = \bigoplus_{n>0} R_n \iff R = R_0[f_1, \dots, f_n]$$

therefore,

R Noetherian $\iff$ $R_0 \subseteq R$ alg-fin