<u>Earlier</u>: R Noetherian ring

$R \subseteq S$ alg-fn $\Rightarrow$ S Noetherian

But

R Noetherian
$\not\Rightarrow$ $R \subseteq S$ alg-fn
$R \subseteq S$ Noetherian

R $\mathbb{N}$-graded $\Rightarrow$ $R_+ = \bigoplus_{n>0} R_n$ is an ideal in R

<u>Special case</u> $R_0 = K$ a field

R_+ is the unique homogeneous maximal ideal in R

<u>Prop</u> R $\mathbb{N}$-graded

$f_1, \dots, f_n \in R$ of degree > 0

$$(f_1, \dots, f_n) = R_+ = \bigoplus_{n>0} R_n \iff R = R_0[f_1, \dots, f_n]$$

therefore,

R Noetherian $\mathbb{N}$ graded ring $\iff$ $R_0 \subseteq R$ alg-fn, R_0 Noetherian

<u>Proof</u> ($\Leftarrow$) $R = R_0[f_1, \dots, f_n]$

$r \in R_+ \Rightarrow r = \underbrace{P(f_1, \dots, f_n)}_{\in (f_1, \dots, f_n)}$ for some $P \in R_0[x_1, \dots, x_n]$ with zero constant term

(each monomial is a multiple of some f_i)

($\Rightarrow$) Take $x \in R_d$. Want to show: $x \in R_0[f_1, \dots, f_n]$.

Induction on d: $d=0 \Rightarrow x \in R_0$ ✓

$d>0$: $x = a_1 f_1 + \dots + a_n f_n \quad a_i \in R$

x, f_i all homogeneous $\Rightarrow$ Can chose a_i homogeneous of degree $\deg x - \deg f_i < \deg x$

$\therefore$ By induction, $a_i \in R_0[f_1, \dots, f_n] \Rightarrow x \in R_0[f_1, \dots, f_n]$.

therefore: $\begin{matrix} R_0 \text{ Noetherian} \\ R_0 \subseteq R \text{ alg fin} \end{matrix} \xrightarrow[\text{Basis}]{\text{Hilbert}} R$ Noetherian

$$\left\{ \begin{array}{l} R \text{ Noetherian} \Rightarrow R_0 \cong R/R_+ \text{ Noetherian} \\ R \text{ Noetherian} \Rightarrow R_+ = (f_1, \dots, f_n) \underset{1^{\text{st}} \text{ part}}{\Rightarrow} R = R_0[f_1, \dots, f_n] \end{array} \right.$$

Another application to invariant rings

R graded ring

G group acting on R by degree-preserving automorphisms $R \xrightarrow{\cdot g} R$

Def $R \xrightarrow{\varphi} S$ ring homomorphism (not necessarily graded)

R is a direct summand of S if φ splits as a map of R-mod:

$$\begin{tikzcd} R \arrow[r, "\varphi"'] & S \arrow[l, bend right, dashed, "\pi"'] \end{tikzcd} \qquad \pi\varphi = id_R$$

Consequence: φ is injective $\rightsquigarrow$ assume $R \subseteq S$

so π is R-linear: $\qquad \pi(rs) = r\pi(s)$

and

$\pi|_R(r) = r$ for all $r \in R$

<u>Note</u> π is a splitting $\iff$ π is R-linear and $\pi(1) = 1$

<u>Philosophy</u> R is a direct summand of a nice ring $\Longrightarrow$ R is nice

<u>Lemma</u> I ideal in R, R direct summand of S $\Longrightarrow$ $IS \cap R = R$

<u>Proof</u> Let π be a splitting

If $r \in IS \cap R$, so

$$r = s_1 f_1 + \cdots + s_n f_n \qquad s_i \in S,\ f_i \in I$$

$$r = \pi(x) = \pi(s_1 f_1 + \cdots + s_n f_n) = f_1 \pi(s_1) + \cdots + f_n \pi(s_n) \in I$$

<u>Prop</u> R direct summand of S, S Noetherian $\Longrightarrow$ R Noetherian

<u>Proof</u> $I_1 \subseteq I_2 \subseteq \cdots$ chain in R

$\rightsquigarrow I_1 S \subseteq I_2 S \subseteq \cdots$ chain in S $\Rightarrow$ stops

$\Rightarrow I_1 = I_1 S \cap R \subseteq I_2 = I_2 S \cap R \subseteq \cdots$ stops $\boxtimes$

Prop k field

$R = k[x_1, \dots, x_d]$

G finite group $\circlearrowright R$ k-linear

char $k \nmid |G|$ (always true in char 0)

then R^G is a direct summand of R

Proof $\rho : R \to R^G$

$$r \mapsto \underbrace{\frac{1}{|G|} \sum_{g \in G} g \cdot r}_{\text{actually in } R^G}$$

Claim ρ is a splitting of $R^G \subseteq R$.

- R^G-linear: $r \in R$, $s \in R^G$

$$\rho(rs) = \frac{1}{|G|} \sum_g g \cdot (rs)$$

$$= \frac{1}{|G|} \sum_g (g \cdot r)(g \cdot s)$$

$$= \frac{1}{|G|} \sum_g s\,(g \cdot r)$$

$$= s \frac{1}{|G|} \sum_g g \cdot r \quad = s \rho(r)$$

• $\rho|_{R^G} = id_R$: $s \in R^G$

$$\rho(s) = \frac{1}{|G|} \sum_g g \cdot s = \frac{1}{|G|} \sum_g s = s$$

<u>thm</u> (Hilbert) k field

$R = k[x_1, \ldots, x_n]$

G acting k-linearly on R

$R^G \subseteq R$ direct summand

then R^G is algebra-finite over k.

<u>Proof</u> R^G is $\mathbb{N}$-graded with $R_0 = k$

$R^G \subseteq R$ direct summand $\Rightarrow R^G$ Noetherian

Noetherian graded $\Longrightarrow R^G$ finitely generated $R_0 = k$-algebra

<u>Fun Fact</u> this proof applies to many <u>infinite</u> groups

$\longrightarrow$ those that are linearly reductive,

which basically means we can define a map like ρ

A little bit of geometry

<u>Question</u> To what extent is a system of polynomial equations determined by its solution set?

<u>Baby example</u>: 1 variable

Over $\mathbb{R}$, z^2+1 has an empty solution set

Over $\mathbb{C}$ or any algebraically closed field,

If $a_1, \dots, a_d$ are the solutions to $f(z)=0$, then

$$f(z) = (z-a_1)^{n_1} \cdots (z-a_d)^{n_d}$$

$\Rightarrow$ f completely determined up to factors

If we ask that f has no repeated factors $\Rightarrow$ (f) unique.

More generally, given a system

$$\begin{cases} f_1 = 0 \\ \vdots \\ f_t = 0 \end{cases} \qquad f_i \in k[z]$$

$z=a$ is a solution $\Leftrightarrow$ $g(a)=0 \quad \forall g \in (f_1, \dots, f_t)$

$R = k[z]$ is a PID $\Rightarrow$ $(f_1, \dots, f_t) = (\gcd(f_1, \dots, f_t))$

Def $\mathbb{A}^d_k := \{(a_1, \dots, a_d) \mid a_i \in k\}$ affine d-space over k

Def $T \subseteq k[x_1, \dots, x_d]$

$$Z_k(T) = Z(T) = \{(a_1, \dots, a_d) \in \mathbb{A}^d_k \mid f(a_1, \dots, a_d) = 0 \text{ for all } f \in T\}$$

Subsets of $\mathbb{A}^d_k$ of this form are called <u>varieties</u>.

A variety is irreducible if it cannot be written as a union of two proper subvarieties.

<u>Warning</u> For some authors, variety $\Rightarrow$ irreducible.

<u>Ex</u>: see pretty pictures, M2 example

Def $X \subseteq \mathbb{A}^d_k$

$$I(X) = \{g \in k[x_1, \dots, x_d] \mid g(a_1, \dots, a_d) = 0 \ \forall \underline{a} \in X\}$$

is an ideal in $k[x_1, \dots, x_d]$ (exercise)