I ideal $\xmapsto{Z}$ $Z(I)$ variety $\xmapsto{I}$ $\mathcal{I}(Z(I))$

how does this relate to I?

<u>Ex</u>: $R = k[x]$, $I_n = (x^n)$ $n \geq 1$

$Z(I_n) = \{0\}$ for all n

What do all these <u>different</u> ideals have in common?

<u>Remark</u> $f \in k[x_1, \dots, x_d]$, $\underline{a} \in \mathbb{A}^d$

$f(\underline{a}) \neq 0 \iff f(\underline{a})$ invertible

$\iff$ $b f(\underline{a}) - 1 = 0$ for some b

$\iff$ $y f(\underline{a}) - 1 = 0$ has a solution

so:

$$\begin{cases} f_1 = 0 \\ \vdots \\ f_m = 0 \end{cases} \text{ and } \begin{cases} g_1 \neq 0 \\ \vdots \\ g_n \neq 0 \end{cases} \text{ has a solution}$$

$$\iff \begin{cases} f_1 = 0 \\ \vdots \\ f_m = 0 \end{cases} \text{ and } \begin{cases} y g_1 - 1 = 0 \\ \vdots \\ y g_n - 1 = 0 \end{cases} \text{ has a solution}$$

$$\iff \begin{cases} f_1 = 0 \\ \vdots \\ f_m = 0 \\ y g_1 \cdots g_n - 1 = 0 \end{cases} \text{ has a solution}$$

<u>thm</u> (Strong Nullstellensatz) $k = \overline{k}$

$R = k[x_1, \dots, x_d]$

$$f \in I(Z(I)) \iff f^n \in I \text{ for some } n$$

<u>Proof</u>

$(\Leftarrow)$ $f^n \in I \Rightarrow f^n(a) = 0$ forall $a \in Z(I)$

$\Downarrow$ k field

$f(a) = 0$ for all $a \in Z(I)$

$\Downarrow$

$f \in I(Z(I))$

$(\Rightarrow)$ $f \in I(Z(I))$

so polynomials in $I = 0 \Longrightarrow f = 0$

thus $\begin{cases} \text{polynomials in } I = 0 \\ f \neq 0 \end{cases}$ has no solutions

$\Rightarrow$ $Z(I + (yf - 1)) = \emptyset$ in $R[y]$

$\xrightarrow[\text{Nullstellensatz}]{\text{weak}}$ $I + (yf - 1) = R[y]$

$\iff$ $1 \in I + (yf - 1)$

If $I = (g_1, \dots, g_m)$,

$$1 = r_0 \cdot (1 - yf) + r_1 g_1 + \dots + r_m g_m$$

$$\Big\downarrow \; y \mapsto \frac{1}{f} \quad \text{in } \mathrm{frac}(R[y])$$

$$1 = r_1\left(\underline{x}, \frac{1}{f}\right) \cdot g_1(x) + \dots + r_m\left(\underline{x}, \frac{1}{f}\right) g_m(x)$$

take the largest negative power of f appearing $\Rightarrow$ clear denominators

$$f^n = s_1 g_1 + \dots + \underbrace{s_m g_m}$$

only on $\underline{x}$ equation in R

$$\Rightarrow f^m \in I$$

Definition the radical of an ideal I is

$$\sqrt{I} = \{ f \in R : f^n \in I \text{ for some } n \}$$

I is a radical ideal if $I = \sqrt{I}$

Ex: the radical of an ideal is an ideal

Example prime ideals are radical

$$f^n \in P \Rightarrow f \in P \text{ or } f^{n-1} \in P \Rightarrow \dots \Rightarrow f \in P$$

R is <u>reduced</u> if $\sqrt{(0)} = (0)$

$\Leftrightarrow$ R has no <u>nilpotent</u> elements $(f^n=0, f\neq 0)$

<u>Exercise</u> R/I reduced $\Leftrightarrow$ $I = \sqrt{I}$

Strong Nullstellensatz says:

$$f \in I(\mathcal{Z}(I)) \quad \Leftrightarrow \quad f \in \sqrt{I}$$

so $\boxed{I(\mathcal{Z}(J)) = \sqrt{J}}$

and $\boxed{\mathcal{Z}(I) = \mathcal{Z}(J) \Leftrightarrow \sqrt{I} = \sqrt{J}}$

X variety in $\mathbb{A}^d_k$ the <u>coordinate ring</u> of X is

$$k[X] := \frac{k[x_1, \dots, x_d]}{I(X)}$$

the algebraic properties of $k[X]$ translate into geometric properties of X.

We can interpret $k[X]$ as the ring of polynomial functions on X

<u>Note</u> Every reduced finitely generated k-algebra is the coordinate ring of some variety

Remark In general $I \cap J \neq IJ$, but $\sqrt{I \cap J} = \sqrt{IJ}$

since $Z(I \cap J) = Z(IJ)$

Remark Even if $k \neq \bar{k}$, $Z_k(J) = Z_k(\sqrt{J})$

$J \subseteq \sqrt{J} \Rightarrow Z_k(\sqrt{J}) \subseteq Z_k(J)$

If $a \in Z_k(J)$ and $f \in \sqrt{J}$, $f^n \in J$ for some n, so

$$f^n(a) = 0 \underset{k \text{ field}}{\Longrightarrow} f(a) = 0$$

$\therefore\ a \in Z_k(J)$

What fails then? $I(Z(J))$ is not necessarily $\sqrt{J}$

eg, $Z_{\mathbb{R}}(x^2+1) = \emptyset$

$$\Downarrow$$

$$I(Z_{\mathbb{R}}(x^2+1)) = I(\emptyset) = \mathbb{R}[x] \neq \sqrt{(x^2+1)} = (x^2+1)$$

What failed here was the Weak Nullstellensatz!

Corollary $R = k[x_1, \ldots, x_d]$, $k = \bar{k}$

$$\{\text{subvarieties of } \mathbb{A}^d_k\} \xleftrightarrow[\text{bijection}]{\text{order reversing}} \{\text{radical ideals in } R\}$$

$$X \xmapsto{\ I\ } \{f \in R : X \subseteq Z_k(f)\}$$

$$Z(I) \longleftarrow\!\mapsfromchar\ I$$

Proof $\quad I(Z(J)) = \sqrt{J}$ for any J

I radical $\Rightarrow I(Z(I)) = \sqrt{I} = I$

Given a variety X, $X = Z(J)$ wlog $J = \sqrt{J}$

$\Rightarrow Z(I(X)) = Z(I(Z(J))) = Z(J) = X$

Lemma $\quad X \subseteq \mathbb{A}^d_k$ irreducible $\Leftrightarrow I(X)$ prime

Proof $\quad (\Leftarrow)$ $X = V_1 \cup V_2$, $V_1, V_2 \subsetneq X$ varieties

so $I(X) \subsetneq I(V_1), I(V_2)$, and $I(X) = I(V_1) \cap I(V_2)$

so $\exists f \in I(V_1), f \notin I(V_2) \quad g \in I(V_2), g \notin I(V_1)$

$$fg \in I(V_1) \cap I(V_2) \Rightarrow fg \in I(X)$$

but $fg \notin I(X)$, so $I(X)$ is <u>not</u> prime.

$(\Rightarrow)$ If $I(X)$ is not prime, let $f, g \notin I(X)$, $fg \in I(X)$.

$$X \subseteq Z(fg) = Z(f) \cup Z(g)$$

$$\Rightarrow X = (Z(f) \cap X) \cup (Z(g) \cap X)$$

$$= \underbrace{Z(I(X) + (f))}_{V_f} \cup \underbrace{Z(I(X) + (g))}_{V_g}$$

$$\left.\begin{array}{l} f \notin \underbrace{I(X)}_{\text{radical}} \Rightarrow V_f \subsetneq X \\ g \notin I(X) \Rightarrow V_g \subsetneq X \end{array}\right\} \Rightarrow X \text{ is reducible}$$

Any variety X can be decomposed into a finite union

$$X = V_1 \cup \cdots \cup V_n$$

where $V_i \subsetneq X$ are all irreducible

We can find this decomposition algebraically:

<u>Def</u> A prime P is a minimal prime of I if

$$\underset{Q \text{ prime}}{I \subseteq Q \subseteq P} \Rightarrow Q = P$$

$$\mathrm{Min}(I) = \{P \text{ minimal prime of } I\}$$

<u>will show</u>: $\sqrt{I} = \bigcap_{\substack{P \supseteq I \\ P \text{ prime}}} P = \bigcap_{P \in \mathrm{Min}(I)} P$

will <u>show</u> R Noetherian $\Rightarrow |\mathrm{Min}(I)| < \infty$

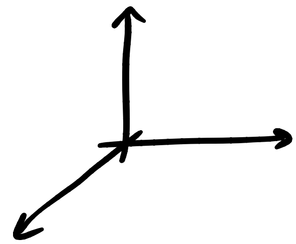
so: X variety

$\Rightarrow I(X) = P_1 \cap \cdots \cap P_k \qquad P_i$ prime

$\Rightarrow X = Z(P_1) \cup \cdots \cup Z(P_k)$

is a decomposition into irreducible varieties

Ex $\quad k[x,y,z]$

$I = (xy, xz, yz) \longleftrightarrow$ 

$\parallel$

$I = (x,y)\cap(y,z)\cap(x,z) \longleftrightarrow \uparrow \cup \swarrow \cup \longrightarrow$

Dictionary	Algebra	$\longleftrightarrow$	Geometry
	radical ideals		varieties
	prime ideals		irreducible varieties
	maximal ideals		points
	(0)		$\mathbb{A}^d$
	$k[x_1, \dots, x_d]$		$\emptyset$
	larger ideals		smaller varieties
	smaller ideals		larger varieties