Varieties $Z(I) \subseteq \mathbb{A}^d$ satisfy:

- $\emptyset, \mathbb{A}^d$ are varieties
- Finite union of varieties is a varieties
- Arbitrary intersection of varieties is a variety

$\Rightarrow$ $Z(I)$ are the closed sets for the $\boxed{\text{Zariski topology}}$ on $\mathbb{A}^d$

(Every variety also inherits the Zariski topology)

Ex: A topological space is Noetherian if every chain $X_1 \supseteq X_2 \supseteq \cdots$ of closed sets stops.
Show that every variety is Noetherian

Ex: Show that if X is a Noetherian topological space then every closed subspace of X is compact

Exercise $\mathbb{A}^d$ is T_1 but not Hausdorff (unless $d=0$)

In fact, a variety with the Zariski topology is never Hausdorff (unless it's finite)

So algebraic geometers say quasicompact for compact (but not Hausdorff)

$\boxed{\text{Spec}}$ R ring

$\mathrm{mSpec}(R) :=$ the set of maximal ideals of R, with the topology with closed sets

$$V_{max}(I) := \{ m \in \mathrm{mSpec}(R) : m \supseteq I \}$$

where I ranges over all ideals in R, including R (so $V(R) = \emptyset$ is closed)

<u>Note</u> By Nullstellensatz, $k = \overline{k}$

$\mathrm{mSpec}(k[x_1, \dots, x_d])$ is homeomorphic to $\mathbb{A}^d_k$

with the Zariski topology

$\mathrm{mSpec}\left(\frac{k[x_1, \dots, x_d]}{I}\right)$ is homeomorphic to $Z_k(I)$

with the Zariski topology

More! this is functorial:

$R \xrightarrow{\varphi} S$ $\rightsquigarrow$ $\mathrm{mSpec}(S) \xrightarrow{\varphi^*} \mathrm{mSpec}(R)$

finitely generated k-algebras

But! this is not the right space to associate to a general R

① Many interesting rings have only <u>one</u> maximal ideal.
the space with 1 element is <u>not</u> exciting

② We would like $R \longmapsto$ some space(R) to be functorial

eg
$$\begin{tikzcd} R = k[x,y] = k[x-1,y] \arrow[d, mapsto] & ? \arrow[d, mapsto] \\ S = k(x)[y] = k(x-1)[y] & (y) \in \mathrm{mSpec}(S) \end{tikzcd}$$

Def the (prime) Spectrum of R is

$\mathrm{Spec}(R) :=$ prime ideals in R, with the topology whose closed sets are

$$V(I) := \{ p \in \mathrm{Spec}(R) \mid p \supseteq I \}$$

where I varies over all the ideals of R, including R.

<u>Example</u> $\mathrm{Spec}(\mathbb{Z})$

$$\begin{tikzcd}[column sep=small] (2) \arrow[drr, no head] & (3) \arrow[dr, no head] & (5) \arrow[d, no head] & (7) \arrow[dl, no head] & \cdots \arrow[dll, no head] \\ & & (0) & & \end{tikzcd}$$

$n \neq 0$

Closed sets: $V((n)) = \{(p) \ni n\} = \{p \mid n\}$

$n = 0 \implies V(0) = \mathrm{Spec}(\mathbb{Z})$

$n = 1 \implies V(\mathbb{Z}) = \emptyset$

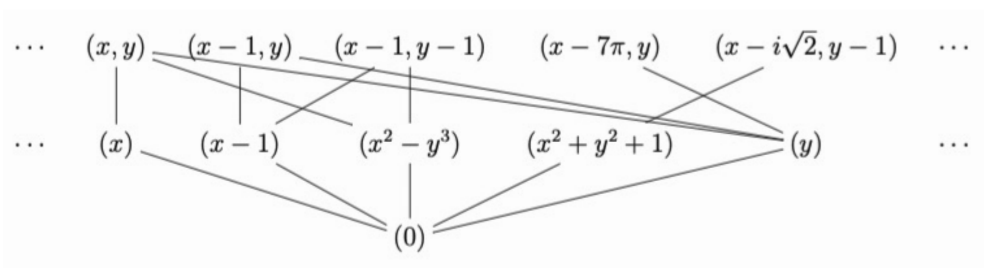
$n \neq 1 \implies$ finite set containing things in the top row

open sets: $\emptyset$, $\mathrm{Spec}(\mathbb{Z})$,

all points but finitely many nonzero ones

Note: Nonempty open sets are dense!

Ex $\mathbb{C}[x,y]$



Prop I, J, I_λ ideals in R (maybe improper)

1) $I \subseteq J \implies V(J) \subseteq V(I)$

2) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$

3) $\bigcap_\lambda V(I_\lambda) = V\left(\sum I_\lambda\right)$

4) $D(f) := \mathrm{Spec}(R) \setminus V(f) = \{p \in \mathrm{Spec}(R) \mid f \notin p\}$
is a basis for the topology on $\mathrm{Spec}(R)$

5) Spec(R) is quasicompact

<u>Proof</u> 4) Open sets are

$$V(I)^c = V(\{f_\lambda\})^c = \bigcap_\lambda V(f_\lambda)^c = \bigcup_\lambda \mathcal{D}(f_\lambda)$$

5) $\emptyset = \bigcap_\lambda V(I_\lambda) = V\left(\sum_\lambda I_\lambda\right)$

$\Rightarrow 1 \in \sum I_\lambda \Rightarrow 1 \in I_{\lambda_1} + \cdots + I_{\lambda_n}$

$\Rightarrow \emptyset = V(I_{\lambda_1} + \cdots + I_{\lambda_n}) = \bigcap_{i=1}^{n} V(I_{\lambda_i})$

$\therefore$ Spec(R) is quasi compact

<u>Def</u> $R \xrightarrow{\varphi} S \rightsquigarrow \mathrm{Spec}(S) \xrightarrow{\varphi^*} \mathrm{Spec}(R)$

$$p \longmapsto \varphi^{-1}(p)$$

so the preimage of a prime ideal by a ring homomorphism is prime

φ^* is:

Notation: $p \cap R := \varphi^*(p)$

- order preserving
- continuous: $\mathcal{U} \subseteq \mathrm{Spec}(R) \Rightarrow \mathcal{U} = V(I)^c$

$q \in (\varphi^*)^{-1}(\mathcal{U}) \Leftrightarrow q \cap R \not\supseteq I \Leftrightarrow q \not\supseteq IS \Leftrightarrow q \in V(IS)$

so $(\varphi^*)^{-1}(\mathcal{U}) = (V(IS))^c$ is open

Ex: $R \xrightarrow{\pi} R/I$ the canonical projection

$\Rightarrow \operatorname{Spec}(R/I) \xrightarrow{\pi^*} \operatorname{Spec}(R)$
is the inclusion

$V(I) \hookrightarrow \operatorname{Spec}(R)$

Def a subset $W \subseteq R$ is **multiplicatively closed** if
- $1 \in W$
- $a, b \in W \Rightarrow ab \in W$

Lemma W multiplicatively closed subset of R
I ideal in R
$W \cap I = \emptyset \Rightarrow$ there exists $\mathfrak{p} \in V(I)$ with $\mathfrak{p} \cap W = \emptyset$

Proof $\mathcal{F} = \{ J \mid J \supseteq I, J \cap W = \emptyset \}$ ordered with $\subseteq$

$I \in \mathcal{F} \Rightarrow \mathcal{F} \neq \emptyset$

$J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots$ chain in $\mathcal{F} \Rightarrow \bigcup J_i \supseteq I, (\bigcup J_i) \cap W = \emptyset$

$\therefore \bigcup J_i \in \mathcal{F}$ is an upper bound for the chain

By Zorn's lemma, $\mathcal{F}$ has a maximal element A
we claim A is prime.

$f, g \notin A \Rightarrow A \subsetneq A+(f), A+(g) \notin \mathcal{F}$

Since $I \subseteq A \subseteq A+(f), A+(g)$, we must have
$(A+(f)) \cap W \neq \emptyset, (A+(g)) \cap W \neq \emptyset$

$x_1 f + a_1 \in W \cap (A + (f))$ $\qquad x_1 \in R,\ a_1 \in A$

$x_2 g + a_2 \in W \cap (A + (g))$ $\qquad x_2 \in R,\ a_2 \in A$

$$\underbrace{\underbrace{(x_1 f + a_1)}_{\in W}\underbrace{(x_2 g + a_2)}_{\in W}}_{\in W} = x_1 x_2 \underbrace{fg}_{\substack{\in A \\ \text{if } fg \in A}} + \underbrace{x_1 f a_2}_{\in A} + \underbrace{x_2 g a_1}_{\in A} + \underbrace{a_1 a_2}_{\in A} \in \underbrace{A \cap W}_{\substack{\overset{\emptyset}{\|} \\ \text{if } fg \in A}}$$

$\therefore\ fg \notin A$ and A is prime $\blacksquare$

<u>Prop</u> $\quad V(I) \subseteq V(f) \iff f \in \sqrt{I}$

Equivalently, $\quad \sqrt{I} = \bigcap_{p \in V(I)} p$

<u>Proof</u> $V(I) \subseteq V(f) \iff f \in p$ for all $p \supseteq I \iff f \in \bigcap_{p \in V(I)} p$

$$\sqrt{I} = \bigcap_{p \in V(I)} p$$

$(\supseteq)$ WTS: $\quad p \supseteq I \Rightarrow p \supseteq \sqrt{I}$

Indeed: $f \in \sqrt{I} \Rightarrow f^n \in I \subseteq p \Rightarrow f \in p$

$(\subseteq)$ $f \notin \sqrt{I}$. Set $W = \{1, f, f^2, \dots\}$. Note: $W \cap \sqrt{I} = \emptyset$

$\Rightarrow$ there exists $p \in V(I)$, $p \cap W = \emptyset \Rightarrow f \notin p$

<u>Corollary</u> There is an order reversing bijection

$$\{\text{closed subsets of } \mathrm{Spec}(R)\} \longleftrightarrow \{\text{radical ideals in } R\}$$