

Problem Set 1

Problem 0. Install Macaulay2.¹

Problem 1. Let $R = \mathbb{Q}[x, y, z]/(x^2, xy)$. Consider the bounded complex

$$\begin{array}{ccccccc}
 C = R & \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} & R^3 & \xrightarrow{\begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}} & R^3 & \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} & R \\
 3 & & 2 & & 1 & & 0
 \end{array}$$

Set C up in Macaulay2 and compute its homology. For which n is $H_n(C) = 0$?

Problem 2. Let $C_n = \mathbb{Z}/8$ for all $n \geq 0$, and $C_n = 0$ for $n < 0$. Let

$$\begin{array}{ccc}
 C_n & \xrightarrow{d_n} & C_{n-1} \\
 x & \longrightarrow & 4x
 \end{array}$$

when $n > 0$, and otherwise let $d_n: C_n \rightarrow C_{n-1}$ be the zero map.

a) Show that (C_\bullet, d_\bullet) is a complex.

b) Compute its homology.

Problem 3 (The Five Lemma). Consider the following commutative diagram of R -modules with exact rows:

$$\begin{array}{ccccccccc}
 A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \\
 a \downarrow & & b \downarrow & & c \downarrow & & d \downarrow & & e \downarrow \\
 A & \longrightarrow & B & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
 \end{array}$$

Show that if $a, b, d,$ and e are isomorphisms, then c is an isomorphism.

Problem 4. Let π denote the canonical projection map from \mathbb{Z} to $\mathbb{Z}/2\mathbb{Z}$. Show that the chain map $f: F \rightarrow G$ given by

$$\begin{array}{ccccccccccc}
 F = & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & \downarrow 0 & & \downarrow 0 & & \downarrow \pi & & \downarrow 0 & & \\
 G = & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & & 2 & & 1 & & 0 & & -1
 \end{array}$$

is a quasi-isomorphism, but not a homotopy equivalence.

¹You can only choose this problem if you did not do the corresponding problem in Commutative Algebra in the Winter.

Problem 5.

- a) Show that in any category, every isomorphism is both an epi and a mono.
- b) Show that the usual inclusion $\mathbb{Z} \rightarrow \mathbb{Q}$ is an epi in the category **Ring**. This *should* feel weird: it says being epi and being surjective are *not* the same thing.
- c) Show that the canonical projection $\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$ is a mono in the category of divisible abelian groups.² Again, this is very strange: it says being monic and being injective are *not* the same thing.

Problem 6. Consider the category $R\text{-mod}$ of R -modules and R -module homomorphisms. Let f be a homomorphism of R -modules. Show that f is injective if and only if it is a monomorphism in $R\text{-mod}$, and that f is surjective if and only if it is an epimorphism in $R\text{-mod}$.

Problem 7. Show that the bijection in the proof of the Yoneda Lemma is natural on both the object and the functor.

Problem 8.

- a) The forgetful functor $\mathbf{Grp} \rightarrow \mathbf{Set}$ is representable.
- b) Given a ring R , the forgetful functor $R\text{-mod} \rightarrow \mathbf{Set}$ is representable.

²An abelian group A is divisible if for every $a \in A$ and every positive integer n there exists $b \in A$ such that $nb = a$.