Problem Set 3

Problem 1. Let R be a ring, I and J ideals in R, and M be an R-module.

a) Show that $R/I \otimes_R R/J \cong R/(I+J)$.

b) Show that $R/I \otimes_R M \cong M/IM$.

c) There is an R-module map $I \otimes_R M \longrightarrow IM$ induced by the R-bilinear map $(a,m) \mapsto am$. This map is always clearly surjective; must it be injective?

Problem 2. Show that over a field k, the Hom functors and the tensor functor are always exact.

Problem 3.

a) Consider R-module homomorphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$. If

$$\operatorname{Hom}_R(M,A) \xrightarrow{f_*} \operatorname{Hom}_R(M,B) \xrightarrow{g_*} \operatorname{Hom}_R(M,C)$$

is exact for all M, then $A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence.[1]

b) Suppose that (F,G) is an adjoint pair of covariant functors R-**mod** $\longrightarrow$ R-**mod**. Show that G is left exact.[2]

Let R be a domain and M be an R-module. The **torsion** of M is the submodule

$$T(M) := \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

The elements of $T(M)$ are called **torsion elements**, and we say that M **is torsion** if $T(M) = M$. Finally, M is **torsion free** if $T(M) = 0$.

Problem 4. Let R be a domain and M be an R-module.

a) The R-module $M/T(M)$ is torsion free.

b) If $f\colon M \longrightarrow N$ is an R-module homomorphism, $f(T(M)) \subseteq T(N)$.

c) Torsion is a left exact covariant functor R-**mod** $\longrightarrow$ R-**mod**.

Problem 5. Let R be a domain with fraction field Q.

a) Show that for every Q-vector space V and every R-module M, $V \otimes_R M \cong V \otimes_R (M/T(M))$.

b) The kernel of the map $M \longrightarrow Q \otimes_R M$ given by $m \mapsto 1 \otimes m$ is $T(M)$.

c) Show that for every Q-vector space V and R-module M, $V \otimes_R M = 0$ if and only if M is torsion.

d) Show that $\mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/(2\pi\mathbb{Z})) \neq 0$.

[1] In particular, we are not assuming $A \xrightarrow{f} B \xrightarrow{g} C$ is a complex!
[2] Later we will explain how it follows easily that F is right exact.

Problem 6. Consider the domain $R = \mathbb{Q}[x, y, z, a, b, c]/(xb-ac, yc-bz, xc-az)$, the ideal $I = (x, a)$ in R, the R-module $N = \mathbb{Q}$, $I = (x, a)$, and consider the 2-generated R-module $M = Rf + Rg$, where the generators f, g satisfy the relations

$$yf - xg = 0 \quad bf - cg = 0 \quad cf - zg = 0.$$

a) Are there nontrivial R-module homomorphisms $M \longrightarrow N$? How about $N \longrightarrow M$?

b) Does $- \otimes_R M$ preserve the injectivity of the inclusion $I \subseteq R$?

c) Apply $\operatorname{Hom}_R(-, R)$ to the short exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0\,.$$

Is R an injective R-module?

Let R be a ring and I be an ideal in R. The functor $\Gamma_I : R\textbf{-mod} \longrightarrow R\textbf{-mod}$ that sends each R-module M to the R-module

$$\Gamma_I(M) := \bigcup_{n \geqslant 1} (0 :_M I^n) = \{m \in M \mid I^n m = 0 \text{ for some } n \geqslant 1\}$$

and that sends each R-module homomorphism $M \xrightarrow{f} N$ to the R-module homomorphism $\Gamma_I(M) \xrightarrow{\Gamma_I(f)} \Gamma_I(N)$ given by restricting the domain and codomain of f is called the ***I*-torsion functor**.

Problem 7.

a) Check that any R-module homomorphism $M \xrightarrow{f} N$ must send $\Gamma_I(M)$ into $\Gamma_I(N)$, so that our definition of the functor Γ_I makes sense.

b) Show that Γ_I is an additive covariant functor.

c) Show that Γ_I is left exact.

d) Show that Γ_I is not necessarily right exact.