A long time ago, in a quarter far far away...

R Noetherian ring $\Leftrightarrow$ every ideal in R is fg

$\Leftrightarrow$ every ascending chain of ideals stops

M Noetherian R-module $\Leftrightarrow$ every submodule of M is fg

$\Leftrightarrow$ every ascending chain of submodules stops

R Noetherian ring : M Noetherian $\Leftrightarrow$ M fg

quotients, submodules of Noetherian modules are Noetherian

Canonical examples $\dfrac{k[x_1, \ldots, x_n]}{I}$ and $\dfrac{k[\![x_1, \ldots, x_n]\!]}{I}$ are Noetherian rings

R ring $\qquad$ M R-module

A prime P is associated to M if

- $P = \mathrm{ann}_R(m)$ for some $m \in M$ $\iff$ • $R/P \hookrightarrow M$

$\mathrm{Ass}(M) :=$ associated primes of M

Facts

- $M \neq 0 \Rightarrow \mathrm{Ass}(M) = \emptyset$
- R Noetherian, M fg $\Rightarrow \mathrm{Ass}(M)$ finite
- Zerodivisors on M $= \displaystyle\bigcup_{P \in \mathrm{Ass}(M)} P$
- $\underbrace{\mathrm{Min}(M)}_{\text{minimal primes}} \subseteq \mathrm{Ass}(M)$

P is a minimal prime of M if P is minimal over $\mathrm{ann}(M)$

- the minimal primes in $\mathrm{Ass}(M)$ are precisely the minimal primes of M

Dimension of R

$$\dim(R) := \sup\{n \mid P_0 \subsetneq \cdots \subsetneq P_n,\ P_i \text{ prime in } R\}$$

$$\operatorname{ht}(P) := \sup\{n \mid P_0 \subsetneq \cdots \subsetneq P_n = P,\ P_i \text{ prime in } R\}$$

$$\operatorname{ht}(I) := \min\{\operatorname{ht}(P) \mid P \in \underbrace{\operatorname{Min}(I)}_{\text{minimal primes containing } I}\}$$

Note: (R, m) local

$\dim(R) = \operatorname{ht}(m)$

Note: minimal primes of R have height 0

Krull's Height theorem $\quad \operatorname{ht}(x_1, \ldots, x_n) \leq n$

$\operatorname{ht}(x) = 1 \iff x$ not in any minimal prime of R

$\dim(k[x_1, \ldots, x_d]) = d$

Idea $\quad X \subseteq \mathbb{A}^n_k$ variety $\longleftrightarrow$ ideal $I(X) \subseteq R = k[x_1, \ldots, x_d]$

$\dim(X) = \dim(R/I)$

geometric idea $\quad$ algebraically defined

$\operatorname{codim}(X) = \operatorname{ht}(I)$

eg: $X =$ 

dim 1

codim $3 - 1 = 2$

$\iff$

$I = (xy, xz, yz)$

$= (x, y) \cap (y, z) \cap (x, z)$

$\operatorname{ht} I = 2$

$\dim(R/I) = 1$