A **complex** of R-modules is a sequence of R-module homomorphisms

$$C = (C_\bullet, d_\bullet) = \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

where $d_n d_{n+1} = 0$ for all n

$\Leftrightarrow$ $\operatorname{im} d_{n+1} \subseteq \ker d_n$

$d_\bullet \equiv$ differentials (of C)

Warning We will not distinguish between complexes and co-complexes

A complex is **exact** at C_n / at n / at homological position n if

$$\operatorname{im} d_{n+1} = \ker d_n$$

exact sequence $\equiv$ exact complex (everywhere)

Homology of $(C_\bullet, d_\bullet)$ is the sequence of R-modules

$$H_n(C) = \frac{\ker d_n}{\operatorname{im} d_{n+1}} = \frac{Z_n(C)}{B_n(C)} \quad \text{nth homology of } C$$

$Z_n(C) := \ker d_n \subseteq C_n$ n-th cycle

$B_n(C) := \operatorname{im} d_{n+1} \subseteq C_n$ nth boundary

Examples

① $$C = \underset{2}{\mathbb{Z}} \xrightarrow{\cdot 4} \underset{1}{\mathbb{Z}} \xrightarrow[\text{projection}]{\overset{\text{canonical}}{\pi}} \underset{0}{\mathbb{Z}/2}$$ (bounded complex)

is a complex: $\mathrm{im}(\cdot 4) = 4\mathbb{Z} \subseteq 2\mathbb{Z} = \ker \pi$

Homology

$H_n(C) = 0$ for $n \geq 3$

$$H_2(C) = \frac{\ker(\mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z})}{\mathrm{im}(0 \longrightarrow \mathbb{Z})} = \frac{0}{0} = 0$$

$$H_1(C) = \frac{\ker(\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2)}{\mathrm{im}(\mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z})} = \frac{2\mathbb{Z}}{4\mathbb{Z}} \cong \mathbb{Z}/2$$

$$H_0(C) = \frac{\ker(\mathbb{Z}/2 \xrightarrow{0} 0)}{\mathrm{im}(\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2)} = \frac{\mathbb{Z}/2}{\mathbb{Z}/2} = 0$$

$H_n(C) = 0$ for $n \leq -1$

exactness at $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \equiv \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z}$ injective

exactness at $\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2 \longrightarrow 0 \equiv \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2$ is surjective

Remark $0 \to A \xrightarrow{f} B$ exact $\Leftrightarrow$ f injective

$B \xrightarrow{g} C \to 0$ exact $\Leftrightarrow$ g surjective

$A \xrightarrow{f} B \xrightarrow{g} C$ exact $\Leftrightarrow$ $\operatorname{im} f = \ker g$

A short exact sequence (ses) is an exact complex

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

so: • A includes in B by f $(A \overset{f}{\subseteq} B)$

• g surjective $\Rightarrow$ $C \cong B/\ker g = B/\operatorname{im} f = \operatorname{coker} f$

so our ses looks like

$$0 \to \ker g \xrightarrow{f} B \xrightarrow{g} \operatorname{coker} f \to 0$$

Remark $0 \to M \to 0$ exact $\Leftrightarrow$ $M = 0$

Remark $0 \to M \xrightarrow{f} N \to 0$ exact $\Leftrightarrow$ f isomorphism.

Ex: $R = k[x]$, k field

$$0 \to R \xrightarrow{\cdot x} R \to R/(x) \to 0 \quad \text{ses.}$$

Category theory

A category $\mathscr{C}$ consists of:

- a collection of **objects**, $ob(\mathscr{C})$
- for each two objects A and B in $\mathscr{C}$,

a collection $Hom_{\mathscr{C}}(a,b)$ of **arrows** from a to b
(a = source, b = target)

- for each A, B, C objects in $\mathscr{C}$, a composition

$$
\begin{aligned}
Hom_{\mathscr{C}}(A,B) \times Hom_{\mathscr{C}}(B,C) &\longrightarrow Hom_{\mathscr{C}}(A,C) \\
f \quad , \quad g &\longmapsto g \circ f = gf
\end{aligned}
$$

satisfying the following axioms:

① $Hom_{\mathscr{C}}(A,B)$ are all disjoint

② For all $A \in ob(\mathscr{C})$, $\exists\, 1_A \in Hom_{\mathscr{C}}(A,A)$ st

$$1_A \circ f = f \qquad g \circ 1_A = g$$

for all $f \in Hom_{\mathscr{C}}(B,A)$ and all $g \in Hom_{\mathscr{C}}(A,B)$

③ Composition is associative: $f \circ (g \circ h) = (f \circ g) \circ h$

$\mathscr{C}$ is **locally small** if $Hom_{\mathscr{C}}(A,B)$ is a set for all objects A, B