Previously, on Homological Algebra:

- $\mathrm{Hom}_R(M,-) : R\text{-mod} \longrightarrow R\text{-mod}$ is left exact
- $\mathrm{Hom}_R(-,M) : R\text{-mod} \longrightarrow R\text{-mod}$ is left exact
- $M \otimes_R - : R\text{-mod} \longrightarrow R\text{-mod}$ is right exact
- $\mathrm{Hom}_R(P,-)$ is exact $\iff$ P is <u>projective</u> $\iff$

$$\begin{tikzcd} & P \arrow[dl, dashed, "g"'] \arrow[d, "f"] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

- $\mathrm{Hom}_R(-,E)$ is exact $\iff$ E is <u>injective</u> $\iff$

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & A \arrow[u, "f"] \arrow[r] & B \arrow[ul, dashed, "g"'] \end{tikzcd}$$

$\iff$

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & I \arrow[u, "f"] \arrow[r] & R \arrow[ul, dashed, "g"'] \end{tikzcd}$$

ideal

<u>today</u> Every module embeds into some injective module.

<u>step 1</u> Abelian groups

An R-module D is <u>divisible</u> if for every $r \in R$, $r \neq 0$ and $d \in D$ there exists $b \in D$ such that $rb = d$.

$\iff$ $M \xrightarrow{\cdot r} M$ is surjective for all $r \neq 0$, $r \in R$.

Example R domain $\Longrightarrow$ $\mathrm{Frac}(R)$ is divisible.

Lemma R domain

Injective $\Longrightarrow$ divisible.

Proof E injective
$r \neq 0,\ r \in R$
$a \in E$

$$\underbrace{\begin{tikzcd} sa \in E \ \leftarrow \text{injective} \\ sr \in (r) \arrow[u] \arrow[u, "f"'] \end{tikzcd}}_{\text{is well-defined because } R \text{ domain}}$$

extends to R

$sr = s'r \Rightarrow s = s'$

$$\Longrightarrow \quad \begin{array}{ccc} R & \xrightarrow{f} & E \\ 1 & \longmapsto & e \end{array} \qquad re = rf(1) = f(r) = 1 \cdot a = a$$

$\square$

the converse is false in general

Lemma R is a PID

a) E is injective $\Longleftrightarrow$ E is divisible

b) Quotients of injective modules are injective

Proof a) $\Rightarrow$ ✓

$\Leftarrow$ E divisible

$$\begin{tikzcd} E \\ I = (a) \arrow[u, "f"] \end{tikzcd} \quad \rightsquigarrow \quad \exists e \quad f(a) = ae \quad \rightsquigarrow \quad \begin{tikzcd} E & re \\ R \arrow[u, "g"] & \ni r \arrow[u, mapsto] \end{tikzcd}$$

ideal $\neq 0$

$$g(ra) = rae = rf(a) = f(ra) \ ✓$$

b) E injective $\Rightarrow$ E divisible $\Rightarrow$ E/D divisible $\Rightarrow$ E/D injective

$\square$

<u>Lemma</u> D injective abelian group
R ring

$\operatorname{Hom}_{\mathbb{Z}}(R, D)$ is an injective R-module

<u>Proof</u> $E := \operatorname{Hom}_{\mathbb{Z}}(R, D)$ is an R-module

$$r \cdot \overset{\cup}{f} := (s \mapsto f(rs))$$

$$\operatorname{Hom}_R(-, \operatorname{Hom}_{\mathbb{Z}}(R, D)) \underset{\text{natural}}{\cong} \operatorname{Hom}_{\mathbb{Z}}(- \otimes_R R, D)$$

$$= \underbrace{\operatorname{Hom}_{\mathbb{Z}}(-, D)}_{\substack{\text{exact} \\ \text{because } D \text{ injective} \\ \text{over } \mathbb{Z}}} \circ \underbrace{(- \otimes_R R)}_{\substack{\underset{\text{natural}}{\cong} \mathrm{Id}_{R\text{-mod}} \\ \text{exact}}} \quad \text{is exact.}$$

<u>theorem</u> Every R-module embeds into an injective R-module.

<u>Proof</u> M R-module $\rightsquigarrow$ as a $\mathbb{Z}$-module,

$$M \cong \bigoplus_i \mathbb{Z}/k \hookrightarrow \bigoplus_i \mathbb{Q}/k \quad \text{injective:}$$

$\mathbb{Q}$ divisible $\mathbb{Z}$-module $\Rightarrow$ injective $\Rightarrow$ $\bigoplus_i \mathbb{Q}$ injective

$M \hookrightarrow$ injective/divisible $\mathbb{Z}$-module D

$\mathrm{Hom}_{\mathbb{Z}}(R,-)$ is left exact $\Rightarrow \mathrm{Hom}_{\mathbb{Z}}(R,M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R,D)$

$$\begin{aligned} M &\xrightarrow{\ \psi\ } \mathrm{Hom}_{\mathbb{Z}}(R,M) \\ m &\longmapsto (r \mapsto m) \end{aligned}$$

R-module map
injective

$$M \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R,M) \hookrightarrow \underbrace{\mathrm{Hom}_{\mathbb{Z}}(R,D)}_{\text{injective } R\text{-module}}$$

$\boxtimes$

In reality: there is such a thing as the smallest injective module M embeds into (the <u>injective hull</u> of M) and over a Noetherian ring it can be computed explicitly.

<u>Finally</u>

<u>thm</u> I is injective if and only if every ses

$$0 \to I \to B \to C \to 0 \quad \text{splits}$$

<u>Proof</u> $(\Rightarrow)$

I injective $\Rightarrow$

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & I \arrow[u, equal] \arrow[r] & B \arrow[ul, dashed] \end{tikzcd}$$

splitting!

($\Leftarrow$) I embeds into some injective E

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "j"] & E \arrow[r] \arrow[l, bend left, dashed, "q"] & \operatorname{coker} j \arrow[r] & 0 \end{tikzcd} \quad \text{splits}$$

$$\begin{tikzcd} & I \arrow[r, "j"'] & E \arrow[l, bend right, "q"'] \\ 0 \arrow[r] & A \arrow[u] \arrow[r] & B \arrow[u, dashed, "h"'] \end{tikzcd}$$

$g := qh$
extends f

Flat modules

M is flat $\Leftrightarrow$ $M \otimes_R -$ is exact

$\Leftrightarrow$ $M \otimes_R -$ preserves injective maps

Lemma $\{M_i\}_i$ flat $\Leftrightarrow$ $\bigoplus_i M_i$ flat

Idea: natural isos $\bigoplus_i (M_i \otimes_R N) \cong \left(\bigoplus_i M_i\right) \otimes_R N$

thm Projective $\Rightarrow$ flat

Proof $R \otimes_R - \underset{\text{nat}}{\cong} \mathrm{Id}_{R\text{-mod}} \Rightarrow R \otimes_R -$ exact $\Rightarrow R$ is flat.

F free $\Rightarrow F \cong \bigoplus_i R \Rightarrow F$ flat

P projective $\Rightarrow P \overset{\oplus}{\hookrightarrow}$ free thus flat $\Rightarrow P$ flat

Torsion submodule of M is

$T(M) := \{m \in M \mid rm = 0 \text{ for some regular element } r \text{ in } R\}$

M is torsion if $T(M) = M$ $\quad$ M is torsion free if $T(M) = 0$

Lemma R domain
M flat $\Rightarrow$ torsion free

Proof $Q = \text{frac}(R)$ is torsion free

$M \otimes_R Q$ is a Q-vector space

$M \otimes_R Q \cong \bigoplus_i Q$ torsion free

$$M \text{ flat} \Longrightarrow 0 \to \underset{\underset{\textstyle M}{\cong}}{M \otimes_R R} \to M \otimes_R Q \quad \text{exact}$$

so $M \cong$ submodule of a torsion free module

$\Rightarrow M$ torsion free

theorem R PID. Flat $\Leftrightarrow$ torsion free

Proof $(\Rightarrow)$ ✓ $\quad$ Now $(\Leftarrow)$:

Step 1 It's sufficient to check all fg submodules are flat
(see notes)

Submodule of torsion free $\Rightarrow$ torsion free

Need to show: fg torsion free modules over a PID are flat.

Structure theorem: $M \cong \bigoplus_i R/I_i$

R/I has torsion unless $I = 0$

$\therefore M \cong \bigoplus_i R \Rightarrow M$ free $\Rightarrow M$ flat.

<u>thm</u> R ring
W multiplicative set

① For every R-module M, there is an iso of $W^{-1}R$-modules

$$W^{-1}R \otimes_R M \cong W^{-1}M$$

Given $M \xrightarrow{\alpha} N$, $W^{-1}R \otimes \alpha$ is sent to $W^{-1}\alpha$.

② $W^{-1}R$ is a flat R-module

③ $W^{-1}(-)$ is exact

<u>Proof</u> ① $W^{-1}R \times M \longrightarrow W^{-1}M$ is R-bilinear

$$\left(\frac{x}{w}, m\right) \longmapsto \frac{xm}{w}$$

Inverse: $\frac{m}{w} \longmapsto \frac{1}{w} \otimes m$. Details: boring.

② Saw this in Commutative Algebra:

α injective $\Rightarrow$ $W^{-1}\alpha$ injective, because

$$0 = W^{-1}\alpha\left(\frac{m}{w}\right) = \frac{\alpha(m)}{n} \Rightarrow u\alpha(m) = 0 \text{ for some } u \in W$$

$$\Rightarrow \alpha(um) = 0 \underset{\alpha \text{ injective}}{\Longrightarrow} um = 0 \Rightarrow \frac{m}{w} = 0$$

③ $W^{-1}R$ is flat $\Rightarrow$ $W^{-1}R \otimes_R -$ is exact $\Rightarrow$ W^{-1} is exact.

$\blacksquare$

Example R domain
$Q = \mathrm{frac}(R)$

What is $Q \otimes_R -$? It is localization at (0) !