A note on homology:

$H_n : Ch(R) \longrightarrow R\text{-mod}$ is an additive functor

But is **not** exact!

$$\underset{\text{exact}}{A \to B \to C} \implies H_n(A) \to H_n(B) \to H_n(C)$$

is a **Complex**

but **not** necessarily exact!

Examples:

$$0 \to A \xrightarrow{f} B$$

$$\begin{tikzcd}
2 & 0 \arrow[r, "0"'] \arrow[d] & 0 \arrow[d] \\
1 & 0 \arrow[r, "0"'] \arrow[d, "0"'] & \mathbb{Z} \arrow[d, equal] \\
0 & \mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[d] \\
-1 & 0 \arrow[r, "0"'] & 0
\end{tikzcd}
\quad \overset{H_0}{\rightsquigarrow} \quad
\begin{tikzcd}
0 \arrow[r] & H_0(A) \arrow[r, "H_0(f)"] & H_0(B) \\
0 \arrow[r] & \underset{\text{not exact!}}{\mathbb{Z}} \arrow[r, "0"] & 0
\end{tikzcd}$$

$$
\begin{tikzcd}
 & B \arrow[r, "g"] & C \arrow[r] & 0 \\
2 & 0 \arrow[r, "0"'] \arrow[d] & 0 \arrow[d] & \\
1 & \mathbb{Z} \arrow[r, equal] \arrow[d, equal] & \mathbb{Z} \arrow[d] & \\
0 & \mathbb{Z} \arrow[r, "0"] \arrow[d] & 0 \arrow[d] & \\
-1 & 0 \arrow[r, "0"'] & 0 &
\end{tikzcd}
\quad \overset{H_1}{\rightsquigarrow} \quad
\begin{tikzcd}
H_1(B) \arrow[r, "H_2(g)"] & H_1(C) \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}
$$

$\underbrace{\qquad}_{\text{not exact}}$

What's really going on:

$$
0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0
$$

$$\rightsquigarrow$$

induces a LES in homology

the connecting homomorphism is not 0 (!)

(in some degrees)

$$
H_{n+1}(C) \xrightarrow{0} H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{0} H_{n-1}(A)
$$

usually not 0

usually not 0

Previously, on Homological Algebra:

→ P projective $\Leftrightarrow$ $\mathrm{Hom}_R(P,-)$ is exact $\Leftrightarrow$
$$\begin{tikzcd} & P \arrow[dl, dotted, "\exists"'] \arrow[d] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$
$\Uparrow$
P free

- Every R-module M is a quotient of a (free $\Rightarrow$) projective module

→ E injective $\Leftrightarrow$ $\mathrm{Hom}_R(-,E)$ is exact $\Leftrightarrow$
$$\begin{tikzcd} & E & \\ 0 \arrow[r] & I \arrow[u] \arrow[r] & R \arrow[ul, dotted, "\exists"'] \end{tikzcd}$$
ideal

- Every R-module M embeds into some injective module.

Projective Resolutions

Slogan: Approximate M by projectives

A projective resolution of M is a complex

$$\cdots \to \overset{2}{P_2} \to \overset{1}{P_1} \to \overset{0}{P_0} \to 0 \to \cdots$$

with $H_i(P_\bullet) = 0$ for $i > 0$ and $H_0(P_\bullet) = M$.

$\Leftrightarrow$ an exact complex $\cdots \to \overset{2}{P_2} \to \overset{1}{P_1} \to \overset{0}{P_0} \to M \to 0$

A free resolution of M is a projective resolution where all the P_i are free.

We sometimes write $P_\bullet \twoheadrightarrow M$ or $P_\bullet \to M$

Remark projective resolution
$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "0"'] & P_1 \arrow[r] \arrow[d, "0"'] & P_0 \arrow[r] \arrow[d, Rightarrow] & 0 \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$
quasi iso

Every module has a free resolution:

Step 1 Find a surjection from a free module $P_0 \xrightarrow{\pi_0} M$

Step 2 Look at the kernel of π_0

Find a free module surjecting onto $K_0 := \ker \pi_0$

$$\begin{tikzcd}
P_1 \arrow[rr, dashed, "i_0 \circ \pi_1"] \arrow[dr, "\pi_1"'] & & P_0 \arrow[r, "\pi_0"] & M \arrow[r] & 0 \\
& K_0 \arrow[ur, "i_0"'] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Note $\ker \pi_1 = \ker (i_0 \circ \pi_1)$

i_0 injective

$$\begin{tikzcd}
0 \arrow[dr] & & 0 & & & & \\
& K_1 \arrow[ur] \arrow[dr] & & & & & \\
P_2 \arrow[ur] \arrow[rr, dashed] & & P_1 \arrow[rr] \arrow[dr] & & P_0 \arrow[r] & M \arrow[r] & 0 \\
& & & K_0 \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Repeat. If $K_i = 0$ at some point, stop

$$\mathrm{pdim}_R(M) := \inf \{ d \mid \exists\, P_\bullet \to M \text{ with } P_i = 0 \text{ for } i > d \}$$

Minimal Free resolution

Setup:

(R, m) Noetherian local ring

or

$\mathbb{N}$-graded k-algebra, $R_0 = k$, $m = \bigoplus_{n\geq 1} R_n$

$\left(\text{so } R = \frac{k[x_1, \dots, x_d]}{I}, \ I \text{ homogeneous}, \ m = (x_1, \dots, x_d)\right)$

M fg (graded) R-module

Note: We can find a free resolution of M where all the P_i are fg

Recall $\mu(M) :=$ minimal # of generators of $M = \dim_k(M/mM)$

Can find a surjection $R^{\mu(M)} \twoheadrightarrow M = Rf_1 + \cdots + Rf_n$

$(r_1, \dots, r_n) \longmapsto r_1 f_1 + \cdots + r_n f_n$

In the graded case we can take all the maps to be graded

$R(-f_1) \oplus \cdots \oplus R(-f_n) \twoheadrightarrow M = Rf_1 + \cdots + Rf_n$ $\qquad \deg(f_i) = d_i$

is a degree graded R-module map

$(R(-s))_t = R_{t-s}$ so R_0 lives in degree s

A minimal free resolution of M is one where each $P_i \cong R^{n_i}$ has n_i the smallest possible. In the graded case, we also ask for the maps in the resolution to be degree preserving. So

$\mu(P_0) = \mu(M), \ \mu(P_1) = \mu(K_0), \ \mu(P_{i+1}) = \mu(K_i)$

Will show: Minimal free resolutions are unique!

Betti numbers $\beta_i(M)$:= rank of F_i in a minimal free resolution

Graded betti numbers: $\beta_{ij}(M)$:= # copies of $R(-j)$ in homological degree i

Betti table has $\beta_{ij}(M)$ in position $(i, i+j)$

Example $R = k[x,y,z]$ $\quad M = R/(xy, xz, yz)$

$$0 \to R^2 \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & x \end{pmatrix}} R^3 \xrightarrow{(xy\ xz\ yz)} R \to M$$

$\beta_1(M) = 3 \quad \beta_2(M) = 2 \quad \beta_0(M) = 1$

Graded resolution:

$$0 \to R(-3)^2 \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & x \end{pmatrix}} R(-2)^3 \xrightarrow{(xy\ xz\ yz)} R \to M$$

degree 1

1+2=3

degree 2

$\beta(M) =$

	0	1	2
0	1		
1		3	2
2			

β_{23}

β_{12}

$\beta_{12}(M) = 3$

$\beta_{23}(M) = 2$

Example $R = k[x,y]$ $\quad M = R/(x^2, xy, y^3)$

$$0 \to \begin{matrix} R(-3) \\ \oplus \\ R(-4) \end{matrix} \xrightarrow{\begin{pmatrix} y & 0 \\ -x & y^2 \\ 0 & x \end{pmatrix}} \begin{matrix} R(-2)^2 \\ \oplus \\ R(-3) \end{matrix} \xrightarrow{(x^2\ xy\ y^3)} R \to M$$

Note: $\begin{pmatrix} 0 \\ y^2 \\ x \end{pmatrix}$ lands in $\begin{matrix} \deg 2 \\ \deg 2 \\ \deg 3 \end{matrix}$ so $\begin{matrix} \deg 2+2 = 4 \\ \deg 1+3 = 4 \end{matrix}$ ✓