

A note on homology:

$H_n: \text{Ch}(R) \rightarrow R\text{-mod}$ is an additive functor

But is not exact!

$A \rightarrow B \rightarrow C \implies H_n(A) \rightarrow H_n(B) \rightarrow H_n(C)$
 exact is a Complex

but not necessarily exact!

Examples:

$$0 \rightarrow A \xrightarrow{f} B$$

$$\begin{array}{cccc}
 2 & 0 & \xrightarrow{0} & 0 \\
 & \downarrow & & \downarrow \\
 1 & 0 & \xrightarrow{0} & \mathbb{Z} \\
 & \downarrow & & \parallel \\
 0 & \mathbb{Z} & \xrightarrow{=} & \mathbb{Z} \\
 & \downarrow & & \downarrow \\
 -1 & 0 & \xrightarrow{0} & 0
 \end{array}$$

$\underbrace{H_0}_{\rightsquigarrow}$

$$0 \rightarrow H_0(A) \xrightarrow{H_0(f)} H_0(B)$$

$$0 \rightarrow \underbrace{\mathbb{Z}} \xrightarrow{0} 0$$

not exact!

$$\begin{array}{ccccccc}
 & B & \xrightarrow{g} & C & \longrightarrow & 0 & \\
 2 & 0 & \xrightarrow{0} & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 1 & \mathbb{Z} & = & \mathbb{Z} & & & \\
 & \parallel & & \downarrow & & & \\
 0 & \mathbb{Z} & \xrightarrow{0} & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 -1 & 0 & \xrightarrow{0} & 0 & & &
 \end{array}
 \quad \xrightarrow{H_1} \quad
 \begin{array}{ccccccc}
 & H_1(B) & \xrightarrow{H_2(g)} & H_1(C) & \longrightarrow & 0 & \\
 & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \\
 & & & \underbrace{\hspace{2cm}} & & & \\
 & & & \text{not exact} & & &
 \end{array}$$

What's really going on!

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$



induces a LES in homology

the connecting homomorphism is not 0 (!)
(in some degrees)

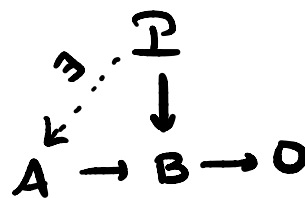
$$\begin{array}{ccccccc}
 & & & & & & \dots(A) \\
 & & & & & \circ & \uparrow \\
 H_{n+1}(C) & \xrightarrow{\circ} & H_n(A) & \longrightarrow & H_n(B) & \longrightarrow & H_n(C) & \longrightarrow & \circ & \longrightarrow & \dots(A) \\
 & \uparrow & & & & & & & \uparrow & & \\
 & \text{usually} & & & & & & & \text{usually} & & \\
 & \text{not 0} & & & & & & & \text{not 0} & &
 \end{array}$$

Previously, on Homological Algebra:

→ P projective $\Leftrightarrow \text{Hom}_R(P, -)$ is exact

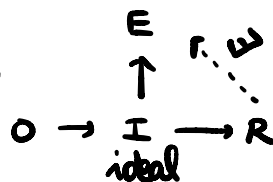
↑
 P free

\Leftrightarrow



• Every R -module M is a quotient of a (free \Rightarrow) projective module

→ E injective $\Leftrightarrow \text{Hom}_R(-, E)$ is exact \Leftrightarrow



• Every R -module M embeds into some injective module

Projective Resolutions

Slogan: Approximate M by projectives

A projective resolution of M is a complex

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow 0 \rightarrow \dots$$

with $H_i(P_\bullet) = 0$ for $i > 0$ and $H_0(P_\bullet) = M$.

\Leftrightarrow an exact complex $\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$

A free resolution of M is a projective resolution where all the P_i are free.

We sometimes write $P_\bullet \twoheadrightarrow M$ or $P_\bullet \rightarrow M$

Remark

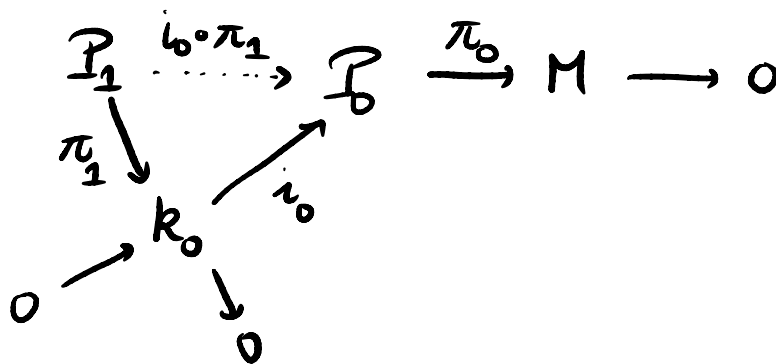
projective resolution

$$\begin{array}{ccccccc} \dots & \rightarrow & P_2 & \rightarrow & P_1 & \rightarrow & P_0 & \rightarrow & 0 \\ & & \circ \downarrow & & \circ \downarrow & & \downarrow & & \text{quasi iso} \\ \dots & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & M & \rightarrow & 0 \end{array}$$

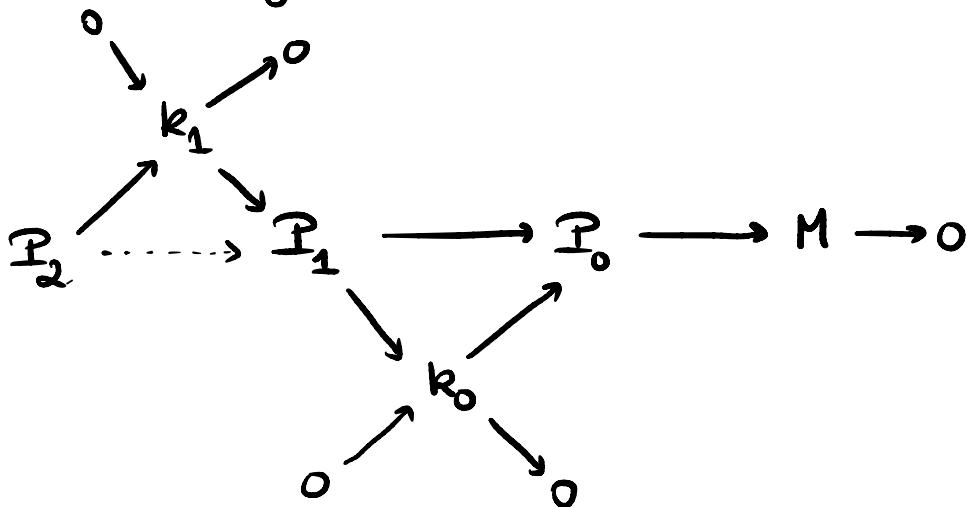
Every module has a free resolution:

Step 1 Find a surjection from a free module $P_0 \xrightarrow{\pi_0} M$

Step 2 Look at the kernel of π_0
Find a free module surjecting onto $k_0 = \ker \pi_0$



Note $\ker \pi_1 = \ker (i_0 \circ \pi_1)$
 \downarrow
 i_0 injective



Repeat. If $k_n = 0$ at some point, stop

$$\text{fdim}_R(M) := \inf \{ d \mid \exists P_i \rightarrow M \text{ with } P_i = 0 \text{ for } i > d \}$$

Minimal Free resolution

Setup:

(R, \mathfrak{m}) Noetherian local ring
or

N -graded k -algebra, $R_0 = k$, $\mathfrak{m} = \bigoplus_{n \geq 1} R_n$

(so $R = \frac{k[x_1, \dots, x_d]}{\mathfrak{I}}$, \mathfrak{I} homogeneous, $\mathfrak{m} = (x_1, \dots, x_d)$)

M fg (graded) R -module

Note: We can find a free resolution of M where all the \mathbb{P}_i are fg

Recall $\mu(M) :=$ minimal # of generators of $M = \dim_k (M/\mathfrak{m}M)$

Can find a surjection $R^{\mu(M)} \rightarrow M = Rf_1 + \dots + Rf_n$
 $(r_1, \dots, r_n) \mapsto r_1f_1 + \dots + r_nf_n$

In the graded case, we can take all the maps to be graded

$R(-f_1) \oplus \dots \oplus R(-f_n) \rightarrow M = Rf_1 + \dots + Rf_n$

is a degree graded R -module map

$\deg(f_i) = d_i$

$(R(-s))_t = R_{t-s}$ so R_0 lives in degree s

A minimal free resolution of M is one where each $\mathbb{P}_i \cong R^{n_i}$ has n_i the smallest possible. In the graded case, we also ask for the maps in the resolution to be degree preserving. So
 $\mu(\mathbb{P}_0) = \mu(M)$, $\mu(\mathbb{P}_1) = \mu(k_0)$, $\mu(\mathbb{P}_{i+1}) = \mu(k_i)$

Will show: Minimal free resolutions are unique!

Betti numbers $\beta_i(M) := \text{rank of } F_i \text{ in a minimal free resolution}$

Graded betti numbers: $\beta_{ij}(M) := \# \text{ copies of } R(-j) \text{ in homological degree } i$

Betti table has $\beta_{ij}(M)$ in position $(i, i+j)$

Example $R = k[x, y, z]$ $M = R/(xy, xz, yz)$

$$0 \rightarrow R^2 \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & x \end{pmatrix}} R^3 \xrightarrow{(xy \ xz \ yz)} R \rightarrow M$$

$$\beta_1(M) = 3 \quad \beta_2(M) = 2 \quad \beta_0(M) = 1$$

Graded resolution:

$$0 \rightarrow R(-3)^2 \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & x \end{pmatrix}} R(-2)^3 \xrightarrow{(xy \ xz \ yz)} R \rightarrow M$$

degree 1 \rightarrow $\begin{pmatrix} z & 0 \\ -y & y \\ 0 & x \end{pmatrix}$ \uparrow degree 2
1+2=3

$$\beta(M) = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & & \\ 1 & & 3 & 2 \\ 2 & & & \end{array}$$

↘ β_{23}
↙ β_{12}

$$\beta_{12}(M) = 3$$

$$\beta_{23}(M) = 2$$

Example $R = k[x, y]$ $M = R/(x^2, xy, y^3)$

$$0 \rightarrow \begin{matrix} R(-3) \\ \oplus \\ R(-4) \end{matrix} \xrightarrow{\begin{pmatrix} y & 0 \\ -x & y^2 \\ 0 & x \end{pmatrix}} \begin{matrix} R(-2)^2 \\ \oplus \\ R(-3) \end{matrix} \xrightarrow{(x^2 \ xy \ y^3)} R \rightarrow M$$

Note: $\begin{pmatrix} 0 \\ y^2 \\ x \end{pmatrix}$ lands in $\begin{matrix} \text{deg } 2 \\ \text{deg } 2 \\ \text{deg } 3 \end{matrix}$ so $\begin{matrix} \text{deg } 2 + 2 = 4 \\ \text{deg } 1 + 3 = 4 \end{matrix} \checkmark$