Minimal Free resolution

Setup:

(R, m) Noetherian local ring

or

$\mathbb{N}$-graded k-algebra, $R_0 = k$, $m = \bigoplus_{n \geq 1} R_n$

$\left(\text{so } R = \frac{k[x_1, \dots, x_d]}{I}, \ I \text{ homogeneous}, \ m = (x_1, \dots, x_d)\right)$

M fg (graded) R-module

A minimal free resolution of M is one where each $F_i \cong R^{n_i}$ has n_i the smallest possible. In the graded case, we also ask for the maps in the resolution to be degree preserving. So
$\mu(F_0) = \mu(M)$, $\mu(F_1) = \mu(K_0)$, $\mu(F_{i+1}) = \mu(K_i)$

Will show: Minimal free resolutions are unique!

Betti numbers $\beta_i(M) :=$ rank of F_i in a minimal free resolution

Graded betti numbers: $\beta_{ij}(M) := \#$copies of $R(-j)$ in homological degree i

Betti table has $\beta_{i,i+j}(M)$ in position (i, j)

	j
i	$\beta_{i,i+j}(M) \longrightarrow (i, j)$

Example $R = k[x,y,z]$ $\qquad M = R/(xy, xz, yz)$

$$0 \to R^2 \xrightarrow[\begin{pmatrix} z & 0 \\ -y & y \\ 0 & x \end{pmatrix}]{} R^3 \xrightarrow{(xy\ \ xz\ \ yz)} R \longrightarrow M$$

$\beta_1(M) = 3 \qquad \beta_2(M) = 2 \qquad \beta_0(M) = 1$

Graded resolution:

$$0 \to R(-3)^2 \xrightarrow[\begin{pmatrix} z & 0 \\ -y & y \\ 0 & x \end{pmatrix}]{} R(-2)^3 \xrightarrow{(xy\ \ xz\ \ yz)} R \longrightarrow M$$

degree 1

$1+2=3$

degree 2

$\beta(M) =$

	0	1	2
0	1		
1		3	2
2			

β_{23}

β_{12}

$\beta_{12}(M) = 3$

$\beta_{23}(M) = 2$

Example $R = k[x,y]$ $\qquad M = R/(x^2, xy, y^3)$

$$0 \to \begin{matrix} R(-3) \\ \oplus \\ R(-4) \end{matrix} \xrightarrow{\begin{pmatrix} y & 0 \\ -x & y^2 \\ 0 & x \end{pmatrix}} \begin{matrix} R(-2)^2 \\ \oplus \\ R(-3) \end{matrix} \xrightarrow{(x^2\ \ xy\ \ y^3)} R \longrightarrow M$$

Note: $\begin{pmatrix} 0 \\ y^2 \\ x \end{pmatrix}$ lands in deg 2, deg 2, deg 3 so deg $2+2=4$ ✓, deg $1+3=4$

betti table

$\beta(M) =$

	0	1	2
0	1	—	—
1	—	2	1
2	—	1	1

$\beta_{00}(M) = 1$

$\beta_{12}(M) = 2$

$\beta_{13}(M) = 1$

$\beta_{23}(M) = 1$

$\beta_{24}(M) = 1$

Equivalently, $F_\bullet$ is a minimal free resolution of M if it is a minimal complex, meaning

$$\cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to 0$$

$\operatorname{im} \partial_i \subseteq \mathfrak{m} F_{i-1}$.

$\iff$ all the entries in the matrices are in $\mathfrak{m}$.

Proof: Suppose $\operatorname{im} \partial_n \subseteq \mathfrak{m}\, \partial_{n-1}$ for all n, but F_n is the free module on non-minimal generators $f_1, \dots, f_s$ generators for $K_{n-1} := \ker \partial_{n-1}$

$$\begin{aligned} F_n &\longrightarrow M \\ e_i &\longmapsto f_i \end{aligned}$$

$f_1, \dots, f_s$ linearly dependent in $M/\mathfrak{m}M$

$\Rightarrow r_1 f_1 + \dots + r_n f_s = 0$ for some $r_i \in R$, not all in $\mathfrak{m}$

Assume wlog that r_s is invertible, and multiply by its inverse

$\Rightarrow e_s - r_1 e_1 - \dots - r_{s-1} e_{s-1} \in \ker \partial_n = \operatorname{im} \partial_{n-1} \subseteq \mathfrak{m} F_n$

$\Rightarrow e_1, \dots, e_s$ are linearly dependent! $\lightning$

Suppose $\mu(F_n) = \mu(K_n)$ for all n, where $K_{n-1} = \ker \partial_{n-1}$, but $\operatorname{im} \partial_{n+1} \not\subseteq \mathfrak{m} F_n$ for some n.

$F_n = \bigoplus_{i=1}^{s} R e_i$, $\{\partial_n(e_i)\}$ minimal generating set for K_{n-1}

$\exists\, r_1 e_1 + \dots + r_s e_s \notin \mathfrak{m} F_n$, $\in \ker \partial_n$ $\overset{\text{WLOG}}{\underset{r_1 \notin \mathfrak{m}}{\Longrightarrow}}$ $e_1 - c_2 e_2 - \dots - c_s e_s \in \ker \partial_n$

$\Rightarrow \partial_n(e_1) = c_2 \partial_n(e_2) + \dots + c_s \partial_n(e_s)$ $\lightning$

So minimal free resolution $\equiv$ take the minimal number of generators needed at each step

<u>Goal</u> Show that there is a unique minimal free resolution (up to isomorphism)

<u>Direct sum of complexes</u>

$$(F, \partial^F) \oplus (G, \partial^G) = \cdots \to \begin{matrix} F_n \\ \oplus \\ G_n \end{matrix} \xrightarrow{\begin{bmatrix} \partial^F & 0 \\ 0 & \partial^G \end{bmatrix}} \begin{matrix} F_{n-1} \\ \oplus \\ G_{n-1} \end{matrix} \to \cdots$$

(this is the coproduct in $Ch(R)$, together with the obvious inclusion maps)

<u>Remark</u> $H_n(F \oplus G) = H_n(F) \oplus H_n(G)$

<u>Remark</u> to check $A = B \oplus C$, need to check:

- $A_n = B_n \oplus C_n$ for all n
- $\partial(B) \subseteq B$, $\partial(C) \subseteq C$

<u>Def</u> A <u>trivial</u> complex is a direct sum of complexes of the form

$$0 \to R \xrightarrow{1} R \to 0$$

<u>Example</u>

$$0 \to R \xrightarrow{\binom{1}{0}} R^2 \xrightarrow{(0\ 1)} R \to 0 \quad \text{is trivial:}$$

$$\begin{array}{ccccccccc} & & & & \| & & & & \\ & & 0 & \to & R & \xrightarrow{1} & R & \to & 0 \\ & & & & \oplus & & & & \\ & & & & 0 & \to & R & \xrightarrow{1} & R & \to 0 \end{array}$$

<u>Lemma</u> $\cdots \to T_2 \xrightarrow{\partial_2} T_1 \xrightarrow{\partial_1} T_0 \to 0$ $\Rightarrow$ trivial

exact everywhere, T_i (graded) fg free modules

<u>Proof</u>

$$0 \to \ker \partial_1 \to T_1 \xrightarrow{\partial_1} \underbrace{T_0}_{\text{projective}} \to 0 \quad \text{splits}$$

$$\Rightarrow \quad T_1 = T_0 \oplus \ker \partial_1, \quad \partial_1 = T_0 \oplus \ker \partial_1 \xrightarrow{(1,0)} T_0$$

so

$$T = \begin{array}{c} \cdots \to T_2 \xrightarrow{\partial_2} \ker \partial_1 = \operatorname{im} \partial_2 \longrightarrow 0 \\ \oplus \\ 0 \longrightarrow T_0 \xrightarrow{1} T_0 \longrightarrow 0 \end{array}$$

still exact!

trivial

Now $\ker \partial_1$ is a direct summand of $T_1 \Rightarrow$ free.

(local case: direct summand of free $\Rightarrow$ projective $\Rightarrow$ free
graded case: direct proof of freeness, see new lemma in notes)

<u>thm</u>

$$\begin{array}{l} I_\bullet = \cdots \to I_1 \xrightarrow{\partial_1} I_0 \xrightarrow{\partial_0} M \longrightarrow 0 \\ \hspace{6.5em} \downarrow f \\ C_\bullet = \cdots \to C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} N \longrightarrow 0 \end{array}$$

- I_i all projective, ∂_0 surjective
- $C_\bullet$ exact

$\Rightarrow$ f lifts to a map of complexes

Any two lifts are homotopic

Proof

projective

$$\begin{tikzcd} P_0 \arrow[r, "\partial_0"] \arrow[d, dashed, "\varphi_0"'] & M \arrow[d, "f"] & \\ C_0 \arrow[r, "\delta_0"'] & N \arrow[r] & 0 \end{tikzcd}$$

$$\delta_0 \varphi_0 (\operatorname{im} \partial_1) \subseteq \delta_0 \varphi_0 (\ker \partial_0) = f \partial_0 (\ker \partial_0) = 0$$

$$\Rightarrow \varphi_0 (\operatorname{im} \partial_1) \subseteq \ker \delta_0$$

Induction:

assume we have $P_{n-1} \xrightarrow{\varphi_{n-1}} C_{n-1}$ with $\varphi_{n-1}(\operatorname{im}(\partial_n)) \subseteq \operatorname{im} \delta_n$

projective

$$\begin{tikzcd} P_n \arrow[r, "\partial_n"] \arrow[d, dashed, "\varphi_n"'] & P_{n-1} \arrow[d, "\varphi_{n-1}"] & \\ C_n \arrow[r, "\delta_n"'] & \operatorname{im} \delta_n \arrow[r] & 0 \end{tikzcd}$$

$$\delta_n \varphi_n (\operatorname{im} \partial_{n+1}) \subseteq \delta_n \varphi_n (\ker \partial_n) = \varphi_{n-1} \partial_n (\ker \partial_n) = 0$$

$$\Rightarrow \varphi_n (\operatorname{im} \partial_{n+1}) \subseteq \ker \delta_n = \operatorname{im} \delta_{n+1} \quad \checkmark$$

Given two such lifts φ, ψ, we want to show they are homotopic.
Notice that $\varphi - \psi$ is a lifting of the 0-map, and a homotopy between φ and ψ is the same as a nullhomotopy for $\varphi - \psi$.
So let φ lift the 0-map, and let's show it's nullhomotopic

$$\begin{tikzcd} P_1 \arrow[r, "\partial_1"] \arrow[d, "\varphi_1"] & P_0 \arrow[r, "\partial_0"] \arrow[d, "\varphi_0"] & M \arrow[d, "0"] \\ C_1 \arrow[r, "\delta_1"'] & C_0 \arrow[r, "\delta_0"'] & N \end{tikzcd} \rightsquigarrow \delta_0\varphi_0 = 0 \Rightarrow \operatorname{im}\varphi_0 \subseteq \ker\delta_0 = \operatorname{im}\delta_1$$

$$\begin{tikzcd} & P_0 \arrow[dl, dashed, "h_0"'] \arrow[d, "\varphi_0"] & \\ C_1 \arrow[r, "\delta_1"'] & \operatorname{im}\delta_1 \arrow[r] & 0 \end{tikzcd}$$

projective

set $h_n = 0$ for $n<0$

$$\Rightarrow \varphi_0 = \delta_1 h_0 + \underbrace{h_{-1}}_{=0}\partial_0$$

and

$$\delta_1(\varphi_1 - h_0\partial_1) \underset{\substack{\downarrow \\ \varphi \text{ map of complexes}}}{=} \varphi_0\partial_1 - \delta_1 h_0 \partial_1 = \underbrace{(\varphi_0 - \delta_1 h_0)}_{=0}\partial_1 = 0$$

$\therefore$ $\operatorname{im}(\varphi_1 - h_0\partial_1) \subseteq \ker\delta_1 = \operatorname{im}\delta_2$

Induction Assume we constructed $h_0, \dots, h_n$ with

- $\varphi_n = h_{n-1}\partial_n + \delta_{n+1}h_n$
- $\operatorname{im}(\varphi_{n+1} - h_n\partial_{n+1}) \subseteq \operatorname{im}\delta_{n+2}$

$$\begin{tikzcd} & P_{n+1} \arrow[dl, dashed, "h_{n+1}"'] \arrow[d, "\varphi_{n+1} - h_n\partial_{n+1}"] \\ C_{n+2} \arrow[r, "\delta_{n+2}"'] & \operatorname{im}\delta_{n+2} \end{tikzcd}$$

projective

$$\delta_{n+2}(\varphi_{n+2} - h_{n+1}\partial_{n+2}) \underset{\substack{\downarrow \\ \varphi \text{ map of} \\ \text{complexes}}}{=} \varphi_{n+1}\partial_{n+2} - \delta_{n+2} h_{n+1}\partial_{n+2}$$

$$= (\varphi_{n+1} - \delta_{n+2} h_{n+1})\partial_{n+2} = h_n \underbrace{\partial_{n+1}\partial_{n+2}}_{=0} = 0$$

$\therefore$ $\operatorname{im}(\varphi_{n+2} - h_{n+1}\partial_{n+2}) \subseteq \ker \delta_{n+2} = \operatorname{im} \delta_{n+3}$ ✓ $\square$

theorem If F is a minimal free resolution for M, every free resolution for M is of the form

$$F \oplus \text{trivial complex}$$

In particular, F is unique up to isomorphism.

Proof G a free resolution of M $\quad (F, \partial)$, $\operatorname{im}\partial \subseteq mF$

there exist complex maps $F \xrightarrow{\varphi} G$, $G \xrightarrow{\psi} F$ lifting $\begin{matrix} M \\ \| \\ M \end{matrix}$

$\Longrightarrow$ $\psi\varphi \simeq \mathrm{id}_F$ with homotopy h

$\mathrm{id} - \psi_n\varphi_n = \partial_{n+1} h_n + h_{n-1}\partial_n \Rightarrow \operatorname{im}(\mathrm{id} - \psi_n\varphi_n) \subseteq m F_n$ for all n.

Claim $\psi_n\varphi_n$ is an isomorphism for all n.

local case A matrix representing $\psi_n\varphi_n$ in some basis

Id-A has entries all in m $\Longrightarrow A = \begin{pmatrix} 1-a_{11} & \cdots & a_{1s} \\ a_{21} & \ddots & \vdots \\ \vdots & & \\ a_{s1} & & 1-a_{ss} \end{pmatrix} \Rightarrow \det A = 1 + \underbrace{a}_{\in m}$

$\therefore$ $\det A$ invertible $\Longrightarrow A$ iso

Graded case $a \notin m \not\Rightarrow a$ invertible, only if a homogeneous!
so we do a little bit more work (see notes)

$\therefore$ $\chi\varphi$ is an iso of complexes $\rightsquigarrow$ inverse $F \xrightarrow{\xi} F$

$$F \xrightarrow{\varphi_n} G \xrightarrow{\chi_n} F \xrightarrow{\xi_n} F = 1_{F_n}$$

$\Rightarrow$ $\xi_n\chi_n$ is a splitting for φ_n

$\therefore$ $G_n = \varphi_n(F_n) \oplus \underbrace{\ker(\xi_n\chi_n)}_{k_n}$ for all n

Claim $G = \varphi(F) \oplus k$ as complexes

Need to check: $\delta(\varphi(F)) \subseteq \varphi(F)$ and $\delta(k) \subseteq k$.

- φ map of complexes $\Rightarrow$ $\delta\varphi(F) = \varphi\partial(F) \subseteq \varphi(F)$ ✓
- For every $a \in k_{n+1}$, $\delta_{n+1}(a) = \varphi(\underbrace{b}_{\in F_n}) + \underbrace{c}_{\in k_n}$

$$b = id_{F_n}(b) = \xi_n\chi_n\varphi_n(b) = \xi_n\chi_n(\varphi_n(b) + \underset{\ker(\xi_n\chi_n)}{c})$$

$$\underset{\text{by def}}{=} \xi_n\chi_n(\delta_{n+1}(a)) \underset{\chi \text{ map of cxs}}{=} \xi_n\delta_{n+1}\chi_n(a) \underset{\xi \text{ map of cxs}}{=} \delta_{n+1}\xi_n\chi_n(\underbrace{a}_{\in k_n}) = 0$$

$\therefore$ $\delta_{n+1}(a) \in k_n$ $\Rightarrow$ $\delta(k) \subseteq k$ $\therefore$ $G \cong F \oplus k$.

<u>Claim</u> k is trivial

- $0 = H_n(G) = H_n(F \oplus k) = \underbrace{H_n(F)}_{=0} \oplus H_n(k) \Longrightarrow H_n(k) = 0$

so k is exact

- $\underset{\text{free}}{G_n} \cong \underset{\text{free}}{F_n} \oplus k_n \Longrightarrow k_n$ free (only because we are in this nice local/graded setting)

$\blacksquare$