Previously, on Homological Algebra:

Every R-module has a projective resolution.

When R is a Noetherian local ring / fg graded k-algebra, every fg (graded) module has a <u>minimal</u> projective resolution

$$\cdots \to R^{\beta_2(M)} \to R^{\beta_1(M)} \to R^{\beta_0(M)} \to M \to 0$$

$\beta_i(M) :=$ betti numbers of M

<u>Graded case</u>: $\cdots \to \bigoplus_j R(-j)^{\beta_{ij}(M)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0j}(M)} \to M \to 0$

(all maps are degree preserving)

$\beta_{ij}(M) :=$ graded betti numbers of M

> Most interesting invariants of M are encoded in its (graded) betti numbers (and sometimes in the differentials in the minimal resolution)

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<u>Injective Resolutions</u>

exact sequence $0 \to M \to E^0 \to E^1 \to \cdots$ all E_i injective

$\Updownarrow$ $\updownarrow$ cocomplexes (differentials go up)

complex $0 \to E^0 \to E^1 \to \cdots$ with $H^0(E) = M$, $H^i(E) = 0$ for $i \neq 0$

<u>Fact</u> Every R-module has an injective resolution

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \operatorname{coker} \partial^0 \arrow[r] \arrow[dr] & 0 \\
0 \arrow[r] & M \arrow[r, "i_0"] & E^0 \arrow[rr, dashed, "\partial^0"] \arrow[dr] & & E^1 \arrow[ul] \arrow[r, dashed, "\partial^1"] & E^2 \arrow[r] & \cdots \\
 & & & \operatorname{coker} i_0 \arrow[ur] \arrow[dr] & \\
 & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

Graded/local case: every module has a minimal injective resolution

Abelian Categories

A category $\mathcal{A}$ is **preadditive** if

- every $\mathrm{Hom}_{\mathcal{A}}(x,y)$ is an abelian group
- for every x, y, z objects in $\mathcal{A}$,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(x,y) \times \mathrm{Hom}_{\mathcal{A}}(z,x) &\xrightarrow{\circ} \mathrm{Hom}_{\mathcal{A}}(z,y) \\ (f,g) &\longmapsto f\circ g \end{aligned}$$

is bilinear, meaning

$$g\circ(f_1+f_2) = g\circ f_1 + g\circ f_2$$

and

$$(g_1+g_2)\circ f = g_1\circ f + g_2\circ f$$

Example R-mod

$\mathcal{A}, \mathcal{B}$ preadditive categories
An additive functor $F : \mathcal{A} \to \mathcal{B}$ is a functor such that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(x,y) &\xrightarrow{F_{x,y}} \mathrm{Hom}_{\mathcal{B}}(F(x),F(y)) \\ f &\longmapsto F(f) \end{aligned}$$

is a homomorphism of abelian groups.

$\mathcal{C}$ category

- An object i is initial in $\mathcal{C}$ if $\mathrm{Hom}_{\mathcal{C}}(i, x)$ is a singleton for all objects x in $\mathcal{C}$

- An object t is terminal in $\mathcal{C}$ if $\mathrm{Hom}_{\mathcal{C}}(x, t)$ is a singleton for every x

- A zero object is an initial and terminal object.

<u>Exercise</u> Initial, terminal, and zero objects, if they exists, are unique up to isomorphism.

<u>Examples</u>

- R-mod: 0 is a zero object

- Ring: $\mathbb{Z}$ is initial, no terminal object

<u>Def</u> $\mathcal{C}$ category with a 0-object, x, y objects the zero arrow $x \xrightarrow{0} y$ is the unique arrow

$$x \longrightarrow 0 \longrightarrow y$$

Remark In a preadditive category, the 0-arrow $x \xrightarrow{0} y$ is the same as the 0 in $\mathrm{Hom}_{\mathcal{A}}(x, y)$

An Additive category $\mathcal{A}$ is a preadditive category such that:

- $\mathcal{A}$ has a zero object
- $\mathcal{A}$ has all finite products

Lemma In any additive category, finite coproducts exists and coincide with products.

Proof x, y two objects $\Rightarrow$ then product exists

$$Z = x \times y$$

$$\begin{tikzcd} & Z \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ x & & y \end{tikzcd}$$

the universal property of the product gives arrows i_1, i_2

$$\begin{tikzcd} & x \arrow[d, dashed, "i_1"] \arrow[ddl, bend right, "1_x"'] \arrow[ddr, bend left, "0"] & \\ & Z \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ x & & y \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[d, dashed, "i_2"] \arrow[ddl, bend right, "0"'] \arrow[ddr, bend left, "1_y"] & \\ & Z \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ x & & y \end{tikzcd}$$

<u>Claim</u> z, i_1, i_2 are a coproduct for x and y

$$\begin{tikzcd} & z \arrow[d, dashed, "\exists!\, h"] & \\ x \arrow[ur, "i_1"] \arrow[r, "f"'] & w & y \arrow[ul, "i_2"'] \arrow[l, "g"] \end{tikzcd}$$

need to show: $\exists!\ h$

<u>Existence</u>: take $h := f\pi_1 + g\pi_2$

$$h i_1 = f\underbrace{\pi_1 i_1}_{1_x} + g\underbrace{\pi_2 i_1}_{0} = f \quad \checkmark$$

$$h i_2 = f\underbrace{\pi_1 i_2}_{0} + g\underbrace{\pi_2 i_2}_{1_y} = g$$

<u>Uniqueness</u> if h' also satisfies $hi_1 = f,\ hi_2 = g$

$$\Rightarrow (h-h')\, i_1 = f - f = 0\ ,\ (h-h')\, i_2 = g - g = 0$$

<u>Sufficient</u>: 0 is the unique arrow satisfying $\psi\, i_1 = 0,\ \psi\, i_2 = 0$

<u>Useful</u>: $i_1\pi_1 + i_2\pi_2 = 1_z$ since

$$\pi_1(i_1\pi_1 + i_2\pi_2) = \underbrace{\pi_1 i_1}_{1_x}\pi_1 + \underbrace{\pi_1 i_2}_{0}\pi_2 = \pi_1 = \pi_1 1_z \quad \checkmark$$

$$\pi_2(i_1\pi_1 + i_2\pi_2) = \underbrace{\pi_2 i_1}_{0}\pi_1 + \underbrace{\pi_2 i_2}_{1_y}\pi_2 = \pi_2 = \pi_2 1_z \quad \checkmark$$

$\psi i_1 = 0$, $\psi i_2 = 0$

$$\Rightarrow \quad \psi = \psi 1_Z = \psi(i_1\pi_1 + i_2\pi_2) = \underbrace{\psi i_1}_{=0}\pi_1 + \underbrace{\psi i_2}_{=0}\pi_2 = 0$$

$\blacksquare$

From now on: in any additive category
we write $A \oplus B$ for the product $\equiv$ coproduct of A and B
(of course $A\oplus B$ is only defined up to isomorphism)

<u>Remark</u> $A \oplus B$ is characterized by the existence of

$$A \xrightarrow{i_A} A\oplus B \qquad A\oplus B \xrightarrow{\pi_A} A$$

$$B \xrightarrow{i_B} A\oplus B \qquad A\oplus B \xrightarrow{\pi_B} B$$

such that

$$\pi_A i_A = id_A, \quad \pi_B i_B = id_B, \quad i_A\pi_A + i_B\pi_B = id_{A\oplus B}$$

<u>theorem</u> F additive functor between additive categories,

① $F(0) = 0$ and $F(x \xrightarrow{0} y) = F(x) \xrightarrow{0} F(y)$

② $F(A\oplus B) = F(A) \oplus F(B)$

Exercise $\mathcal{A}$ additive category

- f mono $\iff (fg = 0 \Rightarrow g = 0)$
- f epi $\iff (gf = 0 \Rightarrow g = 0)$

$\mathcal{A}$ additive category, $x \xrightarrow{f} y$

the **kernel** of f is an arrow $k \xrightarrow{i} x$ such that

- $\left(k \xrightarrow{i} x \xrightarrow{f} y\right) = 0$
- any $z \xrightarrow{g} x$ such that $z \xrightarrow{g} x \xrightarrow{f} y = 0$

factors uniquely through $\ker f$:

$$\begin{tikzcd}
k \arrow[r, "i"] & x \arrow[r, "f"] & y \\
& z \arrow[ul, dashed, "\exists!"] \arrow[u, "g"'] \arrow[ur, "0"'] &
\end{tikzcd}$$

the <u>cokernel</u> of f is an arrow $y \xrightarrow{p} c$ such that

- $\left(x \xrightarrow{f} y \xrightarrow{p} c \right) = 0$
- any $y \xrightarrow{g} z$ such that $x \xrightarrow{f} y \xrightarrow{g} z = 0$

factors uniquely through coker f:

$$\begin{tikzcd}
x \arrow[r] \arrow[dr, "0"'] & y \arrow[r, "p"] \arrow[d, "g"] & c \arrow[dl, dashed, "\exists!"] \\
& z &
\end{tikzcd}$$

<u>Remarks</u> ⓪ If $k \xrightarrow{i} x$ is a kernel for $x \xrightarrow{f} y$, k is the unique object (up to iso) that has the following universal property:
for every object z and every $z \xrightarrow{g} x$, there exists a unique $z \xrightarrow{h} k$ such that $ih = g$.

① Whenever they exist, kernels are mono and cokernels are epi

$$z \underset{g_2}{\overset{g_1}{\rightrightarrows}} \ker f \longrightarrow x \xrightarrow{f} y$$

$$(\ker f) \circ g_1 = (\ker f) \circ g_2 \Rightarrow (\ker f) \circ (g_1 - g_2) = 0$$

Enough to show: $(\ker f) \circ g = 0 \Rightarrow g = 0$

$$\begin{tikzcd}
\ker f \arrow[r] & x \arrow[r, "f"] & y \\
z \arrow[u, "g"] \arrow[ur, "0"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

0 factors uniquely through $\ker f$

g is a factorization of 0 through $\ker f$

$\Rightarrow g = 0$

② kernels and cokernels do not necessarily exist!

In the category of fg R-modules over a non-Noetherian ring,

if I is an infinitely generated ideal,

$\ker\left(R \xrightarrow{\pi} R/I\right) = I$ not in our category!

In fact, $R \xrightarrow{\pi} R/I$ is an epi, but <u>not</u> a cokernel!

so

③ epis may not be cokernels and monos may not be kernels

④ If $\operatorname{coker}\ker f$ and $\ker\operatorname{coker} f$ exist, then there is a canonical arrow $\operatorname{coker}\ker f \longrightarrow \ker\operatorname{coker} f$

$$
\begin{tikzcd}
\ker f \arrow[r] & x \arrow[r, "f"] \arrow[d] & y \arrow[r] & \operatorname{coker} f \\
& \operatorname{coker}\ker f \arrow[r, dashed, "\exists!"'] \arrow[ur, dashed] & \ker\operatorname{coker} f \arrow[u] &
\end{tikzcd}
$$

$f \circ (\ker f) = 0 \implies f$ factors uniquely through $\operatorname{coker} f \dashrightarrow$

$(\operatorname{coker} f) \circ \dashrightarrow \circ \underbrace{\operatorname{coker}\ker f}_{\text{epi}} = 0 \implies (\operatorname{coker} f)\circ f = 0$

$\implies \dashrightarrow$ factors uniquely through $\ker\operatorname{coker} f \dashrightarrow$

An abelian category is an additive category $\mathcal{A}$ such that:

- $\mathcal{A}$ has all kernels and cokernels
- Every mono is the kernel of its cokernel
- Every epi is the cokernel of its kernel
- For every arrow f, the canonical $\operatorname{coker}\ker f \to \ker \operatorname{coker} f$ is an iso

Canonical example R-mod

Unwrapping the definition: $\mathcal{A}$ is abelian if:

- every $\mathrm{Hom}_{\mathcal{A}}(x,y)$ is an abelian group
- composition $\circ$ of arrows is bilinear, meaning
  $g\circ(f_1+f_2) = g\circ f_1 + g\circ f_2$ and $(g_1+g_2)\circ f = g_1\circ f + g_2\circ f$
- $\mathcal{A}$ has a zero object
- $\mathcal{A}$ has all finite products
- $\mathcal{A}$ has all kernels and cokernels
- Every mono is the kernel of its cokernel
- Every epi is the cokernel of its kernel
- For every arrow f, the canonical $\operatorname{coker}\ker f \to \ker \operatorname{coker} f$ is an iso

Remark

In an abelian category, Every arrow factors as $f = \text{mono} \circ \text{epi}$

$$
\begin{tikzcd}
\ker f \arrow[r] & x \arrow[r, "f"] \arrow[d, "\text{epi}"'] & y \arrow[r] & \operatorname{coker} f \\
& \operatorname{coker} \ker f \arrow[r, dashed, "\text{iso}"'] & \ker \operatorname{coker} f \arrow[u, "\text{mono}"'] &
\end{tikzcd}
$$

Image of f is $\operatorname{im} f := \ker \operatorname{coker} f$

Remark the source of $\operatorname{im} f$ is the unique object (up to iso) such that f factors as

$$x \xrightarrow{\text{epi}} \operatorname{im} f \xrightarrow{\text{mono}} y$$

Exercise $\mathcal{A}$ abelian category

f mono $\Leftrightarrow \ker f = 0$ $\qquad$ f epi $\Leftrightarrow \operatorname{coker} f = 0$

Exercise $\mathcal{A}$ abelian category, f mono, g epi

$\ker(fg) = \ker g$ $\qquad$ $\operatorname{coker}(fg) = \operatorname{coker} f$

$\operatorname{im}(fg) = \operatorname{im} f = f$

$\mathcal{A}$ abelian category

A (Chain) complex in $\mathcal{A}$ (C, ∂) or C is a sequence

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

of objects and arrows such that $\partial_{n-1} \circ \partial_n = 0$ for all n

A map of complexes $C \xrightarrow{f} D$ is a sequence of arrows such that

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_n \arrow[r, "\delta_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes

$Ch(\mathcal{A}) :=$ category of (chain) complexes over $\mathcal{A}$ and maps of complexes

Lemma $\mathcal{A}$ abelian $\Rightarrow$ $Ch(\mathcal{A})$ abelian

Sketch:

- $f+g$ computed degreewise: $(f+g)_n := f_n + g_n$
- $0_{Ch(\mathcal{A})} = \cdots \to 0 \to 0 \to 0 \to \cdots$
- $C \times D = \cdots \to C_n \times D_n \xrightarrow{\partial_n^C \times \partial_n^D} C_{n-1} \times D_n \to \cdots$
- $\ker f$ induced by the universal property of ker

$$\begin{tikzcd}
\ker f_{n+1} \arrow[r] \arrow[d, dashed] & C_{n+1} \arrow[r, "f_{n+1}"] \arrow[d] & D_n \arrow[d] \\
\ker f_n \arrow[r] & C_n \arrow[r, "f_n"'] & D_n
\end{tikzcd}$$

- f mono $\Leftrightarrow$ all f_n mono f epi $\Leftrightarrow$ all f_n epi
- $f = 0 \Leftrightarrow$ all $f_n = 0$