

Previously, on Homological Algebra:

Every R -module has a projective resolution.

When R is a Noetherian local ring / fg graded k -algebra, every fg (graded) module has a minimal projective resolution

$$\dots \rightarrow R^{\beta_2(M)} \rightarrow R^{\beta_1(M)} \rightarrow R^{\beta_0(M)} \rightarrow M \rightarrow 0$$

$\beta_i(M) :=$ betti numbers of M

Graded case: $\dots \rightarrow \bigoplus_j R(-j)^{\beta_{i,j}(M)} \rightarrow \dots \rightarrow \bigoplus_j R(-j)^{\beta_{0,j}(M)} \rightarrow M \rightarrow 0$
 (all maps are degree preserving)

$\beta_{i,j}(M) :=$ graded betti numbers of M

Most interesting invariants of M are encoded in its (graded) betti numbers (and sometimes in the differentials in the minimal resolution)

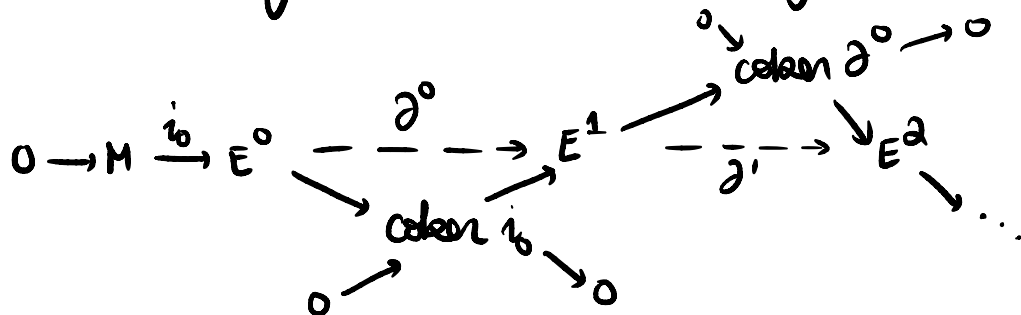
———— // ———— Injective Resolutions

exact sequence $0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$ all E_i injective

\Downarrow \uparrow cocomplexes (differentials go up)

Complex $0 \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$ with $H^0(E) = M, H^i(E) = 0$ for $i \neq 0$

Fact Every R -module has an injective resolution



Graded/local case: every module has a minimal injective resolution

Abelian Categories

A category \mathcal{A} is **preadditive** if

- every $\text{Hom}_{\mathcal{A}}(x, y)$ is an abelian group
- for every x, y, z objects in \mathcal{A} ,

$$\begin{array}{ccc} \text{Hom}_{\mathcal{A}}(x, y) \times \text{Hom}_{\mathcal{A}}(z, x) & \xrightarrow{\circ} & \text{Hom}_{\mathcal{A}}(z, y) \\ (f, g) & \longmapsto & f \circ g \end{array}$$

is bilinear, meaning

$$g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2$$

and

$$(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$$

Example $\mathcal{R}\text{-mod}$

\mathcal{A}, \mathcal{B} preadditive categories

An additive functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is a functor such that

$$\begin{array}{ccc} \text{Hom}_{\mathcal{A}}(x, y) & \xrightarrow{F_{x,y}} & \text{Hom}_{\mathcal{B}}(F(x), F(y)) \\ f & \longmapsto & F(f) \end{array}$$

is a homomorphism of abelian groups.

\mathcal{C} category

- An object i is **initial** in \mathcal{C} if $\text{Hom}_{\mathcal{C}}(i, x)$ is a singleton for all objects x in \mathcal{C}
- An object t is **terminal** in \mathcal{C} if $\text{Hom}_{\mathcal{C}}(x, t)$ is a singleton for every x
- A **zero object** is an initial and terminal object.

Exercise Initial, terminal, and zero objects, if they exist, are unique up to isomorphism.

Examples

- $\mathcal{R}\text{-mod}$: 0 is a zero object
- Ring : \mathbb{Z} is initial, no terminal object

Def \mathcal{C} category with a 0-object, x, y objects
the **zero arrow** $x \xrightarrow{0} y$ is the unique arrow
 $x \rightarrow 0 \rightarrow y$

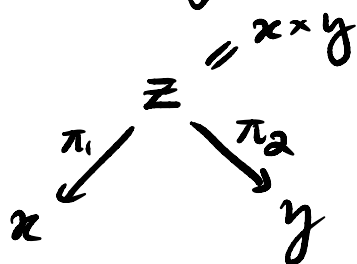
Remark In a preadditive category, the 0-arrow $x \overset{0}{\rightarrow} y$ is the same as the 0 in $\text{Hom}_{\mathcal{A}b}(x, y)$

An Additive category $\mathcal{A}b$ is a preadditive category such that:

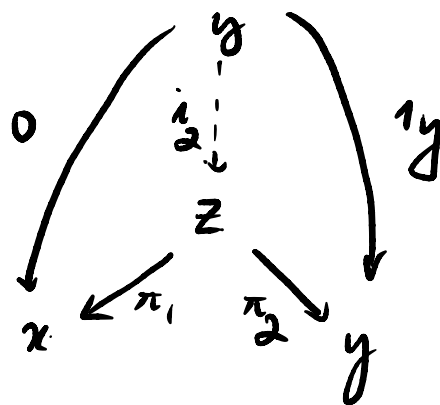
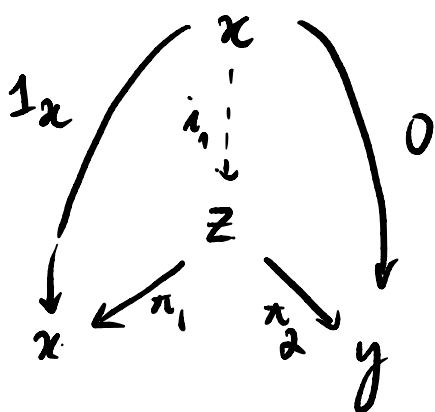
- $\mathcal{A}b$ has a zero object
- $\mathcal{A}b$ has all finite products

Lemma In any additive category, finite coproducts exist and commute with products.

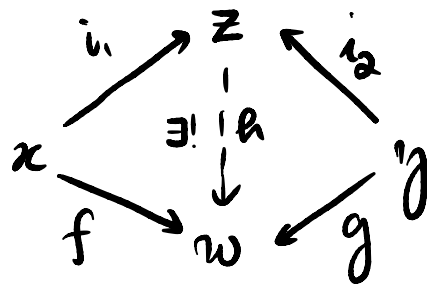
Proof x, y two objects \Rightarrow their product exists



the universal property of the product gives arrows i_1, i_2



Claim z, i_1, i_2 are a coproduct for x and y



need to show:
 $\exists! h$

Existence: take $h := f\pi_1 + g\pi_2$

$$hi_1 = \underbrace{f\pi_1}_{1_x} i_1 + \underbrace{g\pi_2}_0 i_1 = f$$

$$hi_2 = \underbrace{f\pi_1}_0 i_2 + \underbrace{g\pi_2}_{1_y} i_2 = g$$

Uniqueness if h' also satisfies $hi_1=f, hi_2=g$

$$\Rightarrow (h-h')i_1 = f-f=0, (h-h')i_2 = g-g=0$$

Sufficient: 0 is the unique arrow satisfying $\psi i_1=0, \psi i_2=0$

Useful: $i_1\pi_1 + i_2\pi_2 = 1_z$ since

$$\pi_1(i_1\pi_1 + i_2\pi_2) = \underbrace{\pi_1 i_1}_{1_x} \pi_1 + \underbrace{\pi_1 i_2}_0 \pi_2 = \pi_1 = \pi_1 1_z \quad \checkmark$$

$$\pi_2(i_1\pi_1 + i_2\pi_2) = \underbrace{\pi_2 i_1}_0 \pi_1 + \underbrace{\pi_2 i_2}_{1_y} \pi_2 = \pi_2 = \pi_2 1_z \quad \checkmark$$

$$\gamma i_1 = 0, \quad \gamma i_2 = 0$$

$$\Rightarrow \gamma = \gamma 1_2 = \gamma (i_1 \pi_1 + i_2 \pi_2) = \underbrace{\gamma i_1 \pi_1}_{=0} + \underbrace{\gamma i_2 \pi_2}_{=0} = 0 \quad \square$$

From now on: in any additive category
 we write $A \oplus B$ for the product \equiv coproduct of A and B
 (of course $A \oplus B$ is only defined up to isomorphism)

Remark $A \oplus B$ is characterized by the existence of

$$\begin{array}{ccc} A & \xrightarrow{i_A} & A \oplus B & & A \oplus B & \xrightarrow{\pi_A} & A \\ B & \xrightarrow{i_B} & A \oplus B & & A \oplus B & \xrightarrow{\pi_B} & B \end{array}$$

such that

$$\pi_A i_A = \text{id}_A, \quad \pi_B i_B = \text{id}_B, \quad i_A \pi_A + i_B \pi_B = \text{id}_{A \oplus B}$$

theorem F additive functor between additive categories,

$$\textcircled{1} \quad F(0) = 0 \quad \text{and} \quad F(x \xrightarrow{0} y) = F(x) \xrightarrow{0} F(y)$$

$$\textcircled{2} \quad F(A \oplus B) = F(A) \oplus F(B)$$

Exercise Ab additive category

- f mono $\Leftrightarrow (fg=0 \Rightarrow g=0)$
- f epi $\Leftrightarrow (gf=0 \Rightarrow g=0)$

Ab additive category, $x \xrightarrow{f} y$
the kernel of f is an arrow $k \xrightarrow{i} x$ such that

- $(k \xrightarrow{i} x \xrightarrow{f} y) = 0$
- any $z \xrightarrow{g} x$ such that $z \xrightarrow{g} x \xrightarrow{f} y = 0$

factors uniquely through $\ker f$:

$$\begin{array}{ccccc} k & \xrightarrow{i} & x & \xrightarrow{f} & y \\ & \swarrow \exists! & \uparrow g & \searrow 0 & \\ & & z & & \end{array}$$

the cokernel of f is an arrow $y \xrightarrow{p} c$ such that

- $(x \xrightarrow{f} y \xrightarrow{p} c) = 0$

- any $y \xrightarrow{g} z$ such that $x \xrightarrow{f} y \xrightarrow{g} z = 0$

factors uniquely through $\text{coker } f$:

$$\begin{array}{ccccc}
 x & \longrightarrow & y & \xrightarrow{p} & c \\
 & \searrow 0 & \downarrow g & \swarrow \exists! & \\
 & & z & &
 \end{array}$$

Remarks ① If $k \xrightarrow{i} x$ is a kernel for $x \xrightarrow{f} y$, k is the unique object (up to iso) that has the following universal property:

for every object z and every $z \xrightarrow{g} x$, there exists a unique $z \xrightarrow{h} k$ such that $ih = g$.

① Whenever they exist, kernels are mono and cokernels are epi

$$z \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} \text{ker } f \longrightarrow x \xrightarrow{f} y$$

$$(\text{ker } f) \circ g_1 = (\text{ker } f) \circ g_2 \implies (\text{ker } f) \circ (g_1 - g_2) = 0$$

Enough to show: $(\text{ker } f) \circ g = 0 \implies g = 0$

$$\begin{array}{ccccc}
 \text{ker } f & \longrightarrow & x & \xrightarrow{f} & y \\
 g \uparrow & \nearrow 0 & \searrow 0 & \searrow 0 & \\
 z & & & &
 \end{array}$$

0 factors uniquely through $\text{ker } f$

g is a factorization of 0 through $\text{ker } f$
 $\implies g = 0$

② kernels and cokernels do not necessarily exist!

In the category of fg R -modules over a non-Noetherian ring,

if I is an infinitely generated ideal,

$\ker(R \xrightarrow{\pi} R/I) = I$ not in our category!

In fact, $R \xrightarrow{\pi} R/I$ is an epi, but not a cokernel!

so

③ epis may not be cokernels and monos may not be kernels

④ If $\text{coker } \ker f$ and $\ker \text{coker } f$ exist, then

there is a canonical arrow $\text{coker } \ker f \rightarrow \ker \text{coker } f$

$$\begin{array}{ccccc}
 \ker f & \longrightarrow & x & \xrightarrow{f} & y & \longrightarrow & \text{coker } f \\
 & & \downarrow & & \uparrow & & \\
 & & \text{coker } \ker f & \xrightarrow{\exists!} & \ker \text{coker } f & &
 \end{array}$$

$f \circ (\ker f) = 0 \Rightarrow f$ factors uniquely through $\text{coker } f \dashrightarrow$

$(\text{coker } f) \circ \dashrightarrow \circ \underbrace{\text{coker } \ker f}_{\text{epi}} = 0 \Rightarrow (\text{coker } f) \circ f = 0$

$\Rightarrow \dashrightarrow$ factors uniquely through $\ker \text{coker } f \dashrightarrow$

An abelian category is an additive category \mathcal{A}_0 such that:

- \mathcal{A}_0 has all kernels and cokernels
- Every mono is the kernel of its cokernel
- Every epi is the cokernel of its kernel
- For every arrow f , the canonical $\text{coker ker } f \rightarrow \text{ker coker } f$ is an iso

Canonical example $R\text{-mod}$

Unwrapping the definition: \mathcal{A}_0 is abelian if:

- every $\text{Hom}_{\mathcal{A}_0}(x, y)$ is an abelian group
- composition \circ of arrows is bilinear, meaning
 $g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2$ and $(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$
- \mathcal{A}_0 has a zero object
- \mathcal{A}_0 has all finite products
- \mathcal{A}_0 has all kernels and cokernels
- Every mono is the kernel of its cokernel
- Every epi is the cokernel of its kernel
- For every arrow f , the canonical $\text{coker ker } f \rightarrow \text{ker coker } f$ is an iso

Remark

In an abelian category, Every arrow factors as $f = \text{mono} \circ \text{epi}$

$$\begin{array}{ccccc} \text{ker } f & \longrightarrow & x & \xrightarrow{f} & y & \longrightarrow & \text{coker } f \\ & & \text{epi} \downarrow & & \uparrow \text{mono} & & \\ & & \text{coker ker } f & \xrightarrow{\text{iso}} & \text{ker coker } f & & \end{array}$$

Image of f is $\text{im } f := \text{ker coker } f$

Remark the source of $\text{im } f$ is the unique object (up to iso) such that f factors as

$$x \xrightarrow{\text{epi}} \text{im } f \xrightarrow{\text{mono}} y$$

Exercise of abelian category

$$f \text{ mono} \iff \text{ker } f = 0 \quad f \text{ epi} \iff \text{coker } f = 0$$

Exercise of abelian category, f mono, g epi

$$\text{ker}(fg) = \text{ker } g \quad \text{coker}(fg) = \text{coker } f$$

$$\text{im}(fg) = \text{im } f = f$$

\mathcal{A} abelian category

A (chain) complex in \mathcal{A} (C, ∂) or C is a sequence

$$\dots \rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

of objects and arrows such that $\partial_{n-1} \circ \partial_n = 0$ for all n

A map of complexes $C \xrightarrow{f} D$ is a sequence of arrows such that

$$\dots \rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \dots$$

$$f_n \downarrow$$

$$\downarrow f_{n-1}$$

commutes

$$\dots \rightarrow D_n \xrightarrow{\partial_n} D_{n-1} \rightarrow \dots$$

$\text{Ch}(\mathcal{A}) :=$ category of (chain) complexes over \mathcal{A}
and maps of complexes

lemma \mathcal{A} abelian $\Rightarrow \text{Ch}(\mathcal{A})$ abelian

Sketch:

• $f+g$ computed degreewise: $(f+g)_n = f_n + g_n$

• $0_{\text{Ch}(\mathcal{A})} = \dots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \dots$

• $C \times D = \dots \rightarrow C_n \times D_n \xrightarrow{\partial_n^C \times \partial_n^D} C_{n-1} \times D_{n-1} \rightarrow \dots$

• $\ker f$ induced by the universal property of \ker

$$\begin{array}{ccccc} \ker f_{n+1} & \rightarrow & C_{n+1} & \xrightarrow{f_{n+1}} & D_{n+1} \\ \downarrow & & \downarrow & & \downarrow \\ \ker f_n & \rightarrow & C_n & \xrightarrow{f_n} & D_n \end{array}$$

• f mono \Leftrightarrow all f_n mono f epi \Leftrightarrow all f_n epi

• $f=0 \Leftrightarrow$ all $f_n=0$